3ο σετ Ασκησεων / Λυσεις

1. $\int 2(2x+1)^3\,dx$

Θετω $u = 2x+1$, $du = 2dx$ Αρα $\int 2(2x+1)^3dx = \int u^3du = \frac{u^4}{4} + c$

$= \frac{(2x+1)^4}{4} + c$

$$\int x\sin x\,dx = x\cdot(-\cos x) - \int(-\cos x)dx =$$
$$= -x\cdot\cos x + \sin x + c$$

$\int \frac{x^2+2x-1}{x+4}dx$ Κανουμε διαιρεση πολυωνυμων. Αρα

$$\int \frac{x^2+2x-1}{x+4}dx = \int (x-2)dx + \int \frac{7}{x+4}dx = \frac{x^2}{2} - 2x + 7\ln|x+4| + c$$

$\int_{-1}^{2} \frac{dx}{3+2x}$ Θετω $u = 3+2x$, $du = 2dx$, $u_1 = 1$, $u_2 = 7$ Αρα

$$\int_{-1}^{2} \frac{dx}{3+2x} = \int_1^7 \frac{1}{u}\frac{du}{2} = \frac{1}{2}\ln|u|\Big|_1^7 = \frac{1}{2}\ln 7$$

$$\int \frac{dx}{x^2-2x+10}$$

$$x^2-2x+10 = (x^2-2x+1)+9 = (x-1)^2+9$$

Αρα $$\int \frac{dx}{x^2-2x+10} = \int \frac{dx}{(x-1)^2+9} = \frac{1}{3}\,\mathrm{Arctan}\left(\frac{x-1}{3}\right)+c$$

$$\int \frac{d\theta}{\sqrt{27-6\theta-\theta^2}}$$

$$27-6\theta-\theta^2 = -(\theta^2+6\theta+9-36) = -\left((\theta+3)^2-36\right) = 36-(\theta+3)^2$$

Αρα $$\int \frac{d\theta}{\sqrt{27-6\theta-\theta^2}} = \mathrm{Arcsin}\left(\frac{\theta+3}{6}\right)+c$$

2.

$$\int_0^{\frac{\pi}{4}} \cos^5 2x \cdot \sin^2 2x\,dx = \int_0^{\frac{\pi}{4}} \cos 2x\,(1-\sin^2 2x)^2 \cdot \sin^2 2x\,dx$$

Θετω $u=\sin 2x$

$du = 2\cos 2x\,dx$

$u_1=0$

$u_2=1$

$$= \frac{1}{2}\int_0^1 (1-u^2)^2 \cdot u^2\,du = \frac{1}{2}\int_0^1 (u^2-2u^4+u^6)\,du$$

$$= \frac{1}{2}\left(\frac{u^3}{3}-2\frac{u^5}{5}+\frac{u^7}{7}\right)\Big|_0^1 = \frac{4}{105}$$

$$\int \tan^3\theta\,d\theta = \int \tan\theta\,(\sec^2\theta-1)\,d\theta = \int \tan\theta\sec^2\theta\,d\theta - \int \tan\theta\,d\theta$$

Θετω $u=\tan\theta$ και $w=\cos\theta$

$du=\sec^2 d\theta$ $\quad dw=-\sin\theta\,d\theta$

$$= \int u\,du + \int \frac{1}{w}\,dw = \frac{u^2}{2}+\ln|w|+c =$$

$$= \frac{\tan^2\theta}{2}+\ln|\cos\theta|+c$$

$$\left(\sec^2 x = \tan^2 x + 1\right)$$

$$\int \tan^3 x \cdot \sec^4 x \, dx = \int \tan^3 x \cdot \sec^2 x \cdot \sec^2 x \, dx =$$

$$= \int \tan^3 x \left(\tan^2 x + 1\right) \cdot \sec^2 x \, dx$$

Θετω $u = \tan x$

$du = \sec^2 x \, dx$

$$= \int u^3 (u^2+1) \, du = \frac{1}{6} \cdot \tan^6 x + \frac{1}{4} \cdot \tan^4 x + C$$

$$\int \sin^4 x \cdot \cos^2 x \, dx = \int \Big(\underbrace{\frac{1-\cos 2x}{2}}_{\sin^2 x}\Big)^2 \Big(\underbrace{\frac{1+\cos 2x}{2}}_{\cos^2 x}\Big) dx$$

$$= \frac{1}{8} \int \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right) dx =$$

$$= \frac{1}{8} \int \left(1 - \cos 2x - \left(\frac{1+\cos 4x}{2}\right)\right) dx + \frac{1}{8} \int \left(1 - \sin^2 2x\right) \cdot \cos 2x \, dx$$

Χρησιμοποιωντας $u = \sin 2x$ για 2ο ολοκληρωμα και μετα απο απλοποιηση

$du = 2\cos 2x \, dx$

$$= \frac{1}{16} x - \frac{1}{64} \sin 4x - \frac{1}{48} \sin^3 2x + C$$

3.

$$\int \frac{8x+5}{2x^2+3x+1}\,dx$$

$$\frac{8x+5}{2x^2+3x+1} = \frac{8x+5}{2\left(x+\frac{1}{2}\right)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1} = \frac{A(x+1)+B(2x+1)}{(2x+1)(x+1)}$$

$8x+5 = A(x+1) + B(2x+1)$ λυνω και βρισκω $B=3$, $A=2$

Αρα $\dfrac{8x+5}{2x^2+3x+1} = \dfrac{2}{2x+1} + \dfrac{3}{x+1}$

Αρα $$\int \frac{8x+5}{2x^2+3x+1}\,dx = \int \left(\frac{2}{2x+1} + \frac{3}{x+1}\right)dx = \ln\left|(2x+1)(x+1)^3\right| + C$$

$$\int \frac{2x^2+7x+4}{x^3+2x^2+2x}\,dx$$

$$\frac{2x^2+7x+4}{x^3+2x^2+2x} = \frac{2x^2+7x+4}{x(x^2+2x+2)} = \frac{Ax+B}{x^2+2x+2} + \frac{C}{x} =$$

$$= \frac{(Ax+B)x + C(x^2+2x+2)}{(x^2+2x+2)\cdot x}$$

$2x^2+7x+4 = (Ax+B)x + C(x^2+2x+2)$ λυνω και βρισκω $C=2$

$A=0$

$B=3$

Αρα $$\int \frac{2x^2+7x+4}{x^3+2x^2+2x}\,dx = \int \left(\frac{3}{x^2+2x+2} + \frac{2}{x}\right)dx =$$

$$= \int \left(\frac{3}{(x+1)^2+1} + \frac{2}{x}\right)dx = 3\arctan(x+1) + 2\ln|x| + c$$

$$\int \frac{3x^3+4x^2+6x}{(x+2)^2(x^2+4)}\,dx$$

$$\frac{3x^3+4x^2+6x}{(x+2)^2(x^2+4)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+2} + \frac{D}{(x+2)^2} =$$

$$= \frac{(Ax+B)(x+2)^2 + C(x+2)(x^2+4) + D(x^2+4)}{(x^2+4)(x+2)^2}$$

$$3x^3+4x^2+6x = (Ax+B)(x+2)^2 + C(x+2)(x^2+4) + D(x^2+4)$$

Λυνω και $D=-1,\ A=2,\ B=0,\ C=2$

Αρα $\displaystyle\int \frac{3x^3+4x^2+6x}{(x+2)^2(x^2+4)}dx = \int\left(\frac{2x}{x^2+4} + \frac{1}{x+2} - \frac{1}{(x+2)^2}\right)dx =$

$$= \ln|(x^2+4)(x+2)| + \frac{1}{x+2} + C$$

4. ορισοντια αποσταση πτησω απο 0 μεχρι 200 (λυνω $y=0$).

Μεσο-υψος $\displaystyle = \frac{1}{200}\int_0^{200}(2x-0.01x^2)dx = \frac{1}{200}\left(x^2 - \frac{0.01x^3}{3}\right)\Big|_0^{200} =$

↑

Μεση τιμη

$$= 200 - \frac{400}{3} = \frac{200}{3}$$

5. $f'(x) = 3x^2 - \frac{1}{12x^2}$ $\quad f'(x)^2 = 9x^4 - \frac{1}{2} + \frac{1}{144x^4}$

$$L = \int_{\frac{1}{2}}^{2}\sqrt{1+f'(x)^2}\,dx = \int_{\frac{1}{2}}^{2}\sqrt{1+\left(9x^4-\frac{1}{2}+\frac{1}{144x^4}\right)}\,dx$$

$$= \int_{\frac{1}{2}}^{2} \sqrt{\left(3x^2 + \frac{1}{12x^2}\right)^2}\, dx = \int_{\frac{1}{2}}^{2} \left(3x^2 + \frac{1}{12x^2}\right) dx =$$

$$= \left(x^3 - \frac{1}{12x}\right)\Big|_{1/2}^{2} = 8$$

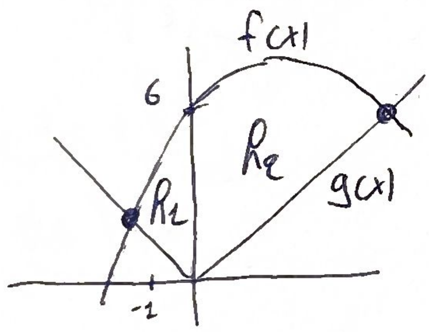
6. Θα εφαρμοσουμε την μεθοδο των δακτυλιων.

Εμβαδον διατομης σε σημειο x: $A(x) = \pi\left(f(x)^2 - g(x)^2\right) =$

$$= \pi\left((\sqrt{x})^2 - (x^2)^2\right) = \pi\left(x - \cancel{4}x^4\right)$$

$$V = \int_0^1 \pi(x - x^4)\,dx = \pi\left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Big|_0^1 = \frac{3\pi}{10}$$

7.



$$g(x) = |2x| = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases}$$

Αριστερο σημειο τομης $f(x), g(x)$: $-2x = -x^2 + 3x + 6$ $\quad x = -1$ (ΔΕΚΤΗ) ή $x = 6$

Δεξι σημειο τομης $2x = -x^2 + 3x + 6$ $\quad x = 3$ (ΔΕΚΤΗ) ή $x = -2$

Αρα $$A = \underbrace{\int_{-1}^{0} \left((-x^2+3x+6) - (-2x)\right) dx}_{R_1} + \underbrace{\int_0^3 \left((-x^2+3x+6) - 2x\right) dx}_{R_2} =$$

$$= \int_{-1}^{0} (-x^2 + 5x + 6)\,dx + \int_0^3 (-x^2 + x + 6)\,dx =$$

$$= \left(-\frac{x^3}{3} + \frac{5}{2}x^2 + 6x\right)\Big|_{-1}^{0} + \left(-\frac{x^3}{3} + \frac{1}{2}x^2 + 6x\right)\Big|_0^3 = \frac{50}{3}$$

8. $f'(x) = \frac{1}{\sqrt{x}}$

$$S = \int_1^3 2\pi f(x)\sqrt{1+f'(x)^2}\,dx = 2\pi\int_1^3 2\sqrt{x}\sqrt{1+\frac{1}{x}}\,dx =$$

$$= 4\pi\int_1^3 \sqrt{x+1}\,dx = \frac{8\pi}{3}(x+1)^{3/2}\Big|_1^3 = \frac{16\pi}{3}(4-\sqrt{2})$$

9. $$a_{n+1} - a_n = \frac{4^{n+1}}{(n+1)^2} - \frac{4^n}{n^2} = \frac{4^{n+1}\cdot n^2 - 4^n(n+1)^2}{n^2(n+1)^2} =$$

$$= \frac{4^n(4n^2-(n+1)^2)}{n^2(n+1)^2} = \frac{4^n(3n^2-2n-1)}{n^2(n+1)^2} = \frac{4^n(n-1)(3n+1)}{n^2(n+1)^2} \geq 0 \ \forall n \in \mathbb{N}^*$$

$$a_{n+1} - a_n = (n+1)^2 + (n+4)^2 - n^2 - (n+3)^2 =$$
$$= n^2+2n+1+n^2+8n+16-n^2-n^2-6n-9 = 4n+8 > 0 \ \forall n \in \mathbb{N}^*$$

10. $$\frac{a_{k+1}}{a_k} = \frac{(k+1)^{k+1}}{(k+1)!}\cdot\frac{k!}{k^k} = \left(\frac{k+1}{k}\right)^k$$

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^k = \lim\left(1+\frac{1}{k}\right)^k = e > 1$$ Αρα Αποκλινει.

$$\frac{a_{k+1}}{a_k} = \frac{((k+1)!)^2}{(2(k+1))!}\cdot\frac{(2k)!}{((k)!)^2} = \frac{(k+1)^2}{(2k+2)(2k+1)}$$

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{(k+1)^2}{(2k+2)(2k+1)} = \frac{1}{4} < 1$$ Αρα συγκλινει

$$\sqrt[k]{\left(\frac{10k^3+k}{9k^3+k+1}\right)^k} = \frac{10k^3+k}{9k^3+k+1}$$

$$\lim_{k\to\infty} \frac{10k^3+k}{9k^3+k+1} = \frac{10}{9} > 1 \text{ Αρα Αποκλινει}$$

$$\sqrt[k]{\left(1+\frac{3}{k}\right)^{k^2}} = \left(1+\frac{3}{k}\right)^k$$

$$\lim_{k\to\infty} \left(1+\frac{3}{k}\right)^k = e^3 > 1 \text{ Αρα αποκλινει}$$