Digital Signal Processing Laboratory
Department of Electrical & Computer Engineering

Digital Signal Processing

Lab 5

Z-transform

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Introduction

The Z-transform is a very powerful mathematical tool for studying discrete signals and systems. It can be used to:

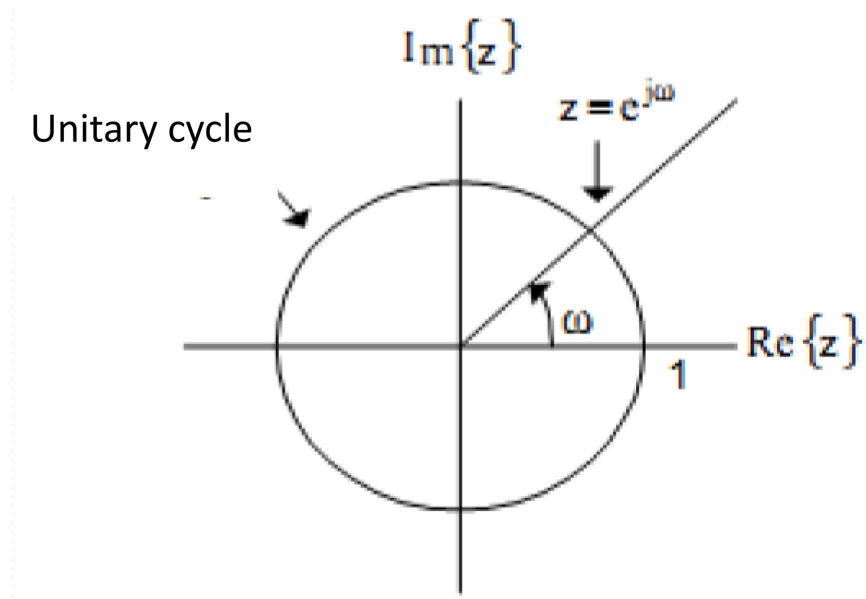
- For solving linear differential equations with constant coefficients.
- In calculating the response of a linear and invariant system to a given input.
- In the design of linear filters.

Introduction

The Z-transfom, of a discrete-time sequence x(n) is defined by the following equation:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

z is a complex number, so in polar form:
Z=Re(z)+jIm(z) = re where r is the measure of z and ω is the angle.



Introduction

The Z-transform, of a discrete-time sequence x(n) is defined by:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

If a sequence x(n) has a Z-transform of X(z), then we write:

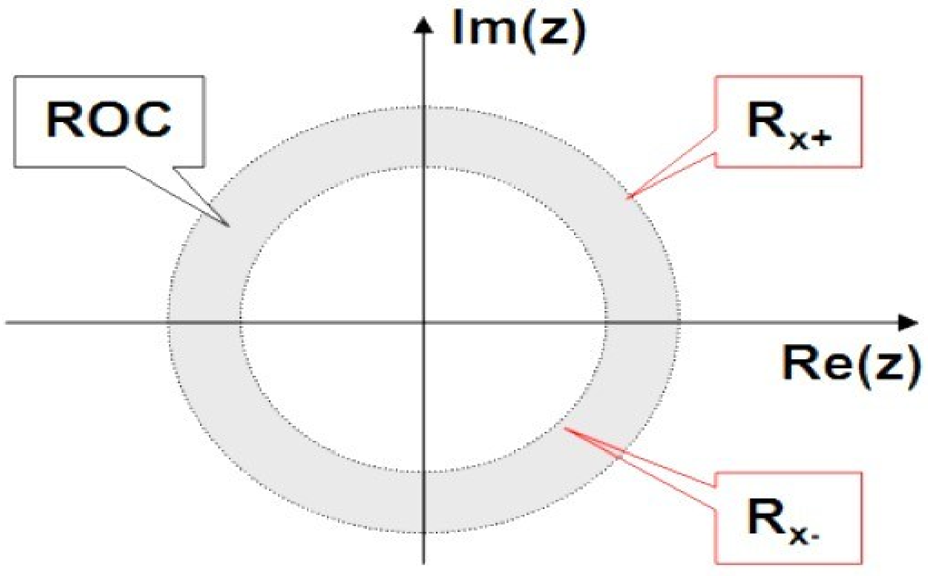
$$x(n) \overset{Z}{\longleftrightarrow} X(z)$$

Convergence region

The set of values of z that X(z) exists define a region in the z plane, called a region of convergence (ROC)

It is defined by two positive numbers R_{x+} και R_{x-} .

$$R_{x-} < |z| < R_{x+}$$



Example 1

Find the Z-transform of the discrete sequence δ(n).

Solution

$$\delta(n) \overset{Z}{\leftrightarrow} X(z) = \sum_{n=0}^{\infty} \delta(n) z^{-n} \Rightarrow$$

$$=> \delta(0)z^{0} + \delta(1)z^{-1} + \delta(2)z^{-2} + \cdots$$

$$=> 1 * 1 + 0 + 0 + \cdots$$

$$= 1$$

Property of Shifting

When shifting a sequence the Z-transform is multiplied by a power of z. That is, if x(n) has a Z-transform of X(z), then:

$$x(n-n_0) \overset{Z}{\longleftrightarrow} z^{-n_0} X(z)$$

For example:

$$\delta(n-n_0) \overset{Z}{\longleftrightarrow} z^{-n_0} X(z) = z^{-n_0} * 1 = z^{-n_0}$$

Example 2

Find the Z-transform of the unit step function u(n).

Solution

$$u(n) \overset{Z}{\leftrightarrow} X(z) = \sum_{n=-\infty}^{\infty} u(n) z^{-n} \Rightarrow$$

$\Rightarrow u(-\infty)z^{-(-\infty)} + \cdots + u(0)z^{-0} + u(1)z^{-1} + \cdots + u(\infty)z^{-\infty}$

$\Rightarrow 0*z^{+\infty} + \cdots + 1*1 + 1*z^{-1} + 1*z^{-2} + 1*z^{-3} \cdots + 1*z^{-\infty}$

$\Rightarrow 1 + z^{-1} + z^{-2} + z^{-3} + \cdots + z^{-\infty}$

$\Rightarrow 1 + (z^{-1})^{1} + (z^{-1})^{2} + (z^{-1})^{3} + \cdots + (z^{-1})^{+\infty}$

$\Rightarrow 1/1\text{-}z^{-1}$

In the previous exercise we used the following identity:

$$\sum_{n=0}^{\infty} Ax^n = \frac{A}{1-x} \qquad |x|< 1$$

Recall the following formula from the geometric series:

$$A + Ax + Ax^2 + \cdots + Ax^{N-1} = \sum_{n=0}^{N-1} Ax^n = \frac{A - Ax^n}{1-x}$$

We can observer that if $|x| < 1$ then $x^n \to 0$ while $N \to \infty$ so we obtain our original equation.

Example 3

Find the Z-transform of the following sequence

$$x(n) = \{\underset{\uparrow}{1}, 0.8, 0.64, 0.512, \dots\}$$

The arrow denotes the time t=0

Solution

$$x(n) \overset{Z}{\leftrightarrow} X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \Rightarrow$$

$$X(z) = x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots$$

$$= 1*1 + 0.8z^{-1} + 0.64z^{-2} + 0.512z^{-3} + \cdots$$

$$= 1 + (0.8z^{-1})^1 + (0.8z^{-1})^2 + (0.8z^{-1})^3 + \cdots$$

$$= 1/1-0.8z^{-1}$$

Z-transform table

Z-transforms of known sequences:

Sequence	Z-transform	Region of Convergence
$\delta(n)$	1	All the values of z
$\delta(n-n_0)$	z^{-n0}	All the values of z, except z=0 If $n_o>0$
$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$

Z-transform

The Z transformation as an explicit function of z:

$$X(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{q} b(k) z^{-k}}{\sum_{k=0}^{p} a(k) z^{-k}} = C \frac{\prod_{k=1}^{q} (1 - \beta_k z^{-1})}{\prod_{k=1}^{p} (1 - \alpha_k z^{-1})}$$

The roots of the numerator β_k are referred as zeros while the roots of the denominator α_k are referred as poles.

Poles and zeros provide a short representation of X(z) which is often represented graphically by the pole-zero diagrams.

The positions of the poles are denoted by 'x' and the positions of the zeros by 'o'. The convergence region is denoted by the shading of the corresponding region in the complex plane -z.

The tf2zpk function

[z, p, k] = scipy.signal.tf2zpk*(num, den)*

Convert transfer functions to poles-and-zero representations. Returns the zeros and poles of the system defined by num/den. k is a gain associated with the system zeros.

Note You should use `tf2zp` when working with positive powers ($s^2 + s + 1$), such as in continuous-time transfer functions. A similar function, `tf2zpk`, is more useful when working with transfer functions expressed in inverse powers ($1 + z^{-1} + z^{-2}$), which is how transfer functions are usually expressed in DSP.

$$H(s) = \frac{B(s)}{A(s)} = \frac{b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{a_1 s^{m-1} + \cdots + a_{m-1} s + a_m}$$

The vector a specifies the coefficients of the denominator polynomial $A(s)$ (or $A(z)$) in descending powers of s (z^{-1}).

Example 4

Find the zeros, poles and coefficient C of the following transfer function, using Python functions.

$$X(z) = \frac{2z^2 + 3z}{z^2 + 0.4z + 1}$$

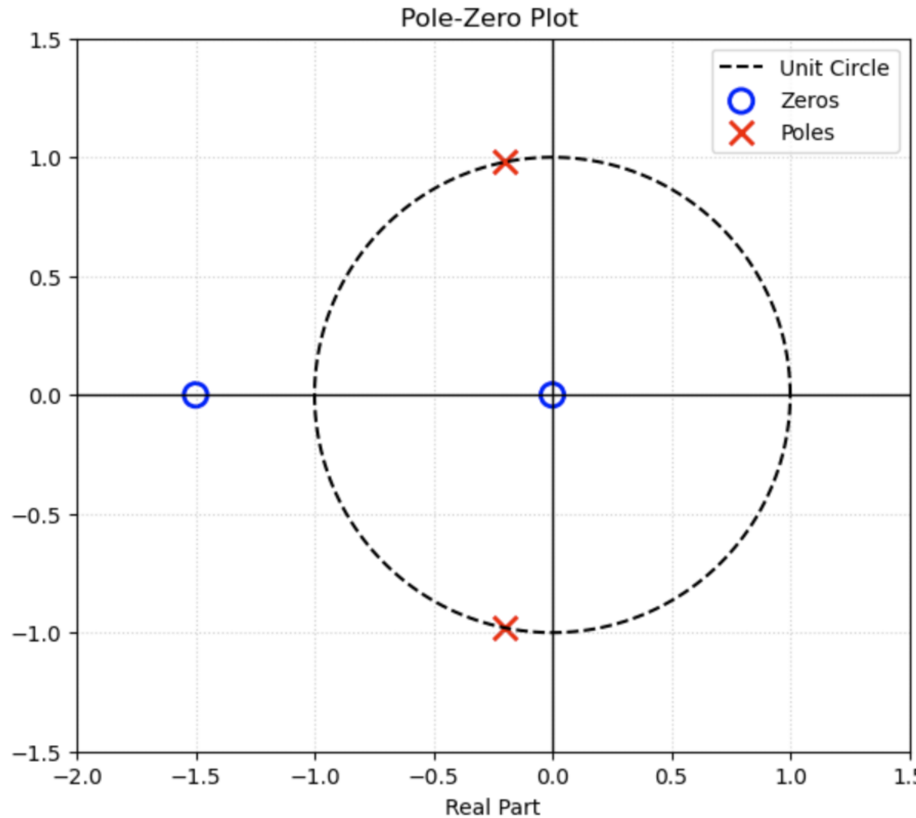
Solution

```
# Calculations
zeros, poles, k = signal.tf2zpk(num, den)

fig, ax = plt.subplots(figsize=(7, 7))

# Plot the Unit Circle
```

```
theta = np.linspace(0, 2*np.pi, 200)
ax.plot(np.cos(theta), np.sin(theta), color='black', linestyle='--', linewidth=1.5, label='Unit Circle')

# Plotting Poles (x) and Zeros (o)
ax.scatter(np.real(zeros), np.imag(zeros), s=120, marker='o', facecolors='none', edgecolors='blue', linewidth=2, label='Zeros')
ax.scatter(np.real(poles), np.imag(poles), s=120, marker='x', color='red', linewidth=2, label='Poles')

# Axis and display settings
ax.axhline(0, color='black', lw=1)
ax.axvline(0, color='black', lw=1)
```

```
ax.set_xlim([-2.0, 1.5])
ax.set_ylim([-1.5, 1.5])
ax.set_aspect('equal')
ax.grid(True, linestyle=':', alpha=0.6)

plt.title(r'Pole-Zero Plot ', fontsize=12)
```

```
plt.xlabel('Real Part')
plt.ylabel('Imaginary Part')
plt.legend()
plt.show()
```



Exercise 5

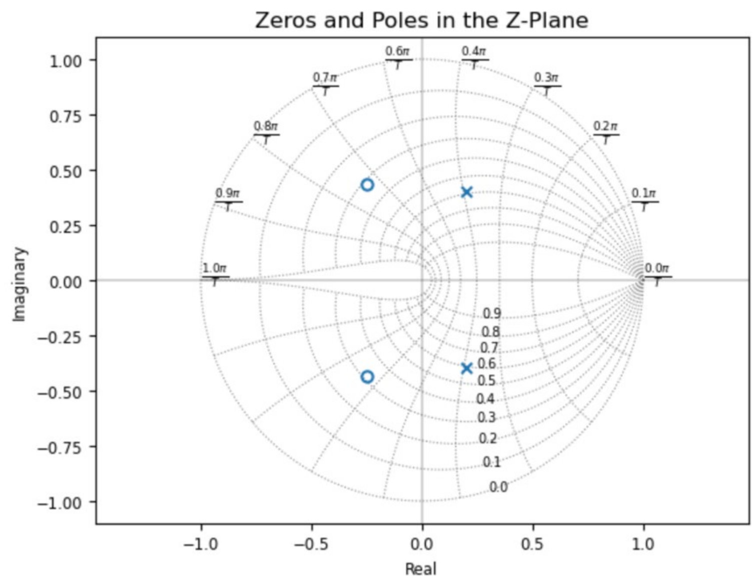
Find the zeros and poles of the following transfer function, using Python functions.

$$H(z) = \frac{1 + 0.5z^{-1} + 0.25z^{-2}}{1 - 0.4z^{-1} + 0.2z^{-2}}$$

Solution #1

```
import numpy as np
import matplotlib.pyplot as plt
import control as ctrl
# Define the transfer function
numerator = [1, 0.5, 0.25]
```

```
denominator = [1, -0.4, 0.2]
system = ctrl.TransferFunction(numerator, denominator,
True)
# Plot the zeros and poles. pzmap
#cannot compute the gain
poles, zeros = ctrl.pzmap(system,
plot=True, grid=True)

# Print the poles and zeros
print("Poles:", poles)
print("Zeros:", zeros)
plt.title('Zeros and Poles in the Z-Plane')
plt.show()
```

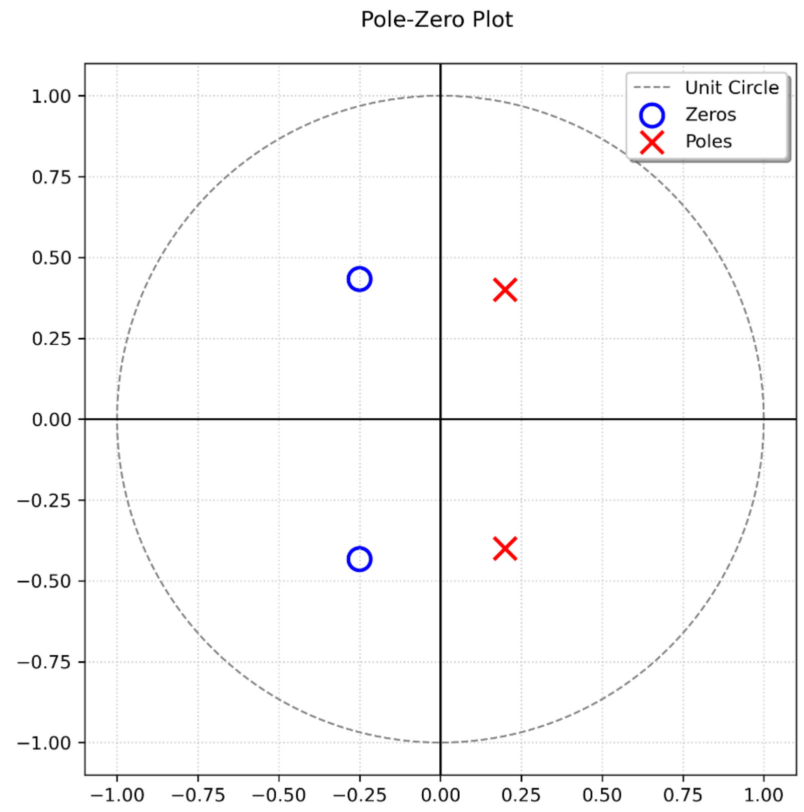


Solution #2

```
import numpy as np
import matplotlib.pyplot as plt
from scipy import signal

# Coefficients
num = [1, 0.5, 0.25]
den = [1, -0.4, 0.2]

# Pole/Zero calculation
zeros, poles, k = signal.tf2zpk(num, den)
plt.figure(figsize=(6, 6), facecolor='white')

# Plotting the Unit Circle
```

```
theta = np.linspace(0, 2*np.pi, 200)
plt.plot(np.cos(theta), np.sin(theta), color='gray', linestyle='--',
label='Unit Circle')

# Plotting the points
plt.scatter(np.real(zeros), np.imag(zeros), s=120, marker='o',
facecolors='none', edgecolors='blue', linewidth=2, label='Zeros')
plt.scatter(np.real(poles), np.imag(poles), s=120, marker='x',
color='red', linewidth=2, label='Poles')
plt.axhline(0, color='black', lw=1)
plt.axvline(0, color='black', lw=1)

plt.xlim([-1.1, 1.1])
plt.ylim([-1.1, 1.1])
plt.gca().set_aspect('equal')
```

```
plt.grid(True, linestyle=':', alpha=0.5)
plt.title(r'Pole – zero plot', fontsize=12)
plt.legend()
plt.show()
```



Example 6

Find the Z-transform of the following sequence:

x(n)= 3δ(n)+ δ(n-2)+ δ(n+2)

Solution

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-2}^{2} x(n) z^{-n}$$

$$= x(-2)z^{2} + x(-1)z^{1} + x(0)z^{0} + x(1)z^{-1} + x(2)z^{-2}$$

$$= z^{2} + 0 + 3 + 0 + z^{-2}$$

$$= z^{2} + 3 + z^{-2}$$

We have already calculated that x(-2)=1, x(-1)=0, x(0)=3, x(1)=0, x(2)=1 .

Example 7

Find the Z-transform of the following sequence:

$$\mathbf{x(n)= 2\delta(n)+ 5\delta(n-3)+ \delta(n+1)}$$

Solution

$X(z)=\sum_{-1}^{3} x(n)z^{-n} = x(-1)z+x(0)z^{0}+x(1)z^{-1}+x(2)z^{-2}+x(3)z^{-3}$

$= 1z+2+0+0+5z^{-3} = z+2+5z^{-3}$

Example 8

Find the Z-transform of the following sequence:

x(n)= 5u(n)

Solution

$$X(z)=\sum_{n=-\infty}^{\infty}5u(n)z^{-n}=5\frac{1}{1-z^{-1}}$$

Example 9

Determine the region of convergence of the following transfer function.

$$H(z) = \frac{z}{z - 0.5}$$

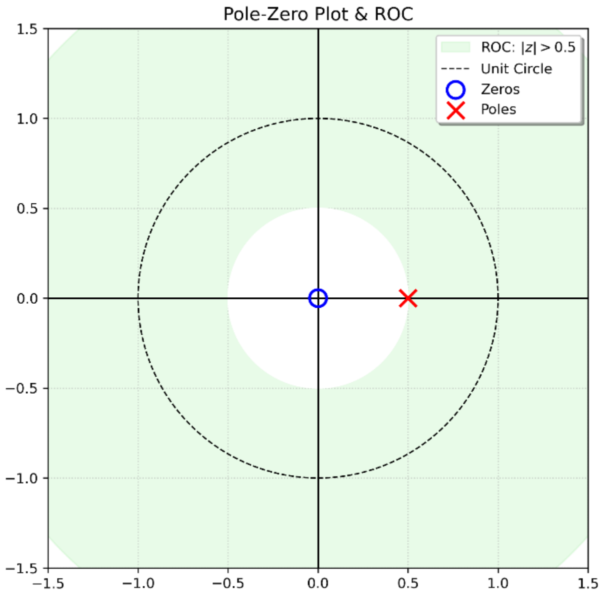
Solution

```
import numpy as np
import matplotlib.pyplot as plt
from scipy import signal

num = [1, 0]
den = [1, -0.5]
```

```
# Pole and Zero calculation
zeros, poles, gain = signal.tf2zpk(num, den)
fig, ax = plt.subplots(figsize=(7, 7), facecolor='white')

# ROC Plotting - Causal System
# Plotting the ROC background
roc_area = plt.Circle((0, 0), 2.0, color='lightgreen', alpha=0.2,
label='ROC: $|z| > 0.5$')
ax.add_artist(roc_area)
# Central “hole” up to the pole
white_center = plt.Circle((0, 0), np.max(np.abs(poles)),
color='white', zorder=2)
ax.add_artist(white_center)
# Plotting the Unit Circle
```

```
theta = np.linspace(0, 2*np.pi, 200)
ax.plot(np.cos(theta), np.sin(theta), color='black', linestyle='--',
linewidth=1, label='Unit Circle')

# Plotting Poles and Zeros
ax.scatter(np.real(zeros), np.imag(zeros), s=150, marker='o',
facecolors='none',
        edgecolors='blue', linewidth=2, label='Zeros', zorder=3)
ax.scatter(np.real(poles), np.imag(poles), s=150, marker='x',
color='red',
        linewidth=2, label='Poles', zorder=3)

# Axis Settings & Legend
ax.axhline(0, color='black', lw=1.2)
ax.axvline(0, color='black', lw=1.2)
```

```
ax.set_xlim([-1.5, 1.5])
ax.set_ylim([-1.5, 1.5])
ax.set_aspect('equal')
ax.grid(True, linestyle=':', alpha=0.6)
# Legend Top Right
ax.legend(loc='upper right', frameon=True, shadow=True)
plt.title(r'Pole-Zero Plot & ROC using $tf2zpk$', fontsize=13)
plt.show()
```



Tip

We can define the following plot_z_plane function and call it for each example.

```
def plot_z_plane(zeros, poles, show_roc=1):
    fig, ax = plt.subplots(figsize=(7, 7), facecolor='white')

    # Maximum pole for ROC calculation and axis limits
    max_p = np.max(np.abs(poles)) if len(poles) > 0 else 0

    # ROC Plotting (Only if show_roc == 1)
    if show_roc == 1:
        roc_limit = max(2.5, max_p + 1.0)
        # Green area
```

```
        roc_area = plt.Circle((0, 0), roc_limit, color='lightgreen', alpha=0.2, label=f'ROC: $|z| > {max_p:.2f}$', zorder=1)
        ax.add_artist(roc_area)
        # White "hole" (on top of ROC, below the X markers)
        white_center = plt.Circle((0, 0), max_p, color='white', zorder=2)
        ax.add_artist(white_center)

    # Unit Circle
    theta = np.linspace(0, 2*np.pi, 200)
    ax.plot(np.cos(theta), np.sin(theta), color='black', linestyle='--',
            linewidth=1, label='Unit Circle', zorder=3)

    # Poles and Zeros
    ax.scatter(np.real(zeros), np.imag(zeros), s=150, marker='o',
```

```
        facecolors='none', edgecolors='blue', linewidth=2,
        label='Zeros', zorder=4)
ax.scatter(np.real(poles), np.imag(poles), s=150, marker='x',
        color='red', linewidth=2, label='Poles', zorder=4)
# Automatic axis adjustment
all_coords = np.concatenate(([1.2], np.abs(poles), np.abs(zeros)))
limit = np.max(all_coords) + 0.3
ax.set_xlim([-limit, limit])
ax.set_ylim([-limit, limit])
  ax.axhline(0, color='black', lw=1, zorder=3)
ax.axvline(0, color='black', lw=1, zorder=3)
ax.set_aspect('equal')
ax.legend(loc='upper right', frameon=True).set_zorder(5)
ax.grid(True, linestyle=':', alpha=0.5, zorder=0)
plt.show()
```

Example 10

Determine the region of convergence of the following transfer function.

$$G(z) = \frac{z}{3z^2 - 4z + 1}$$

Solution

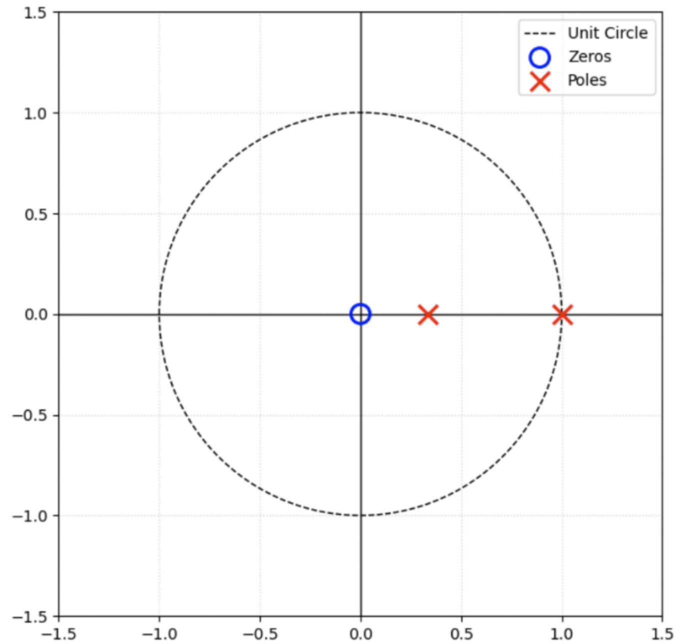
b = [0, 1, 0]

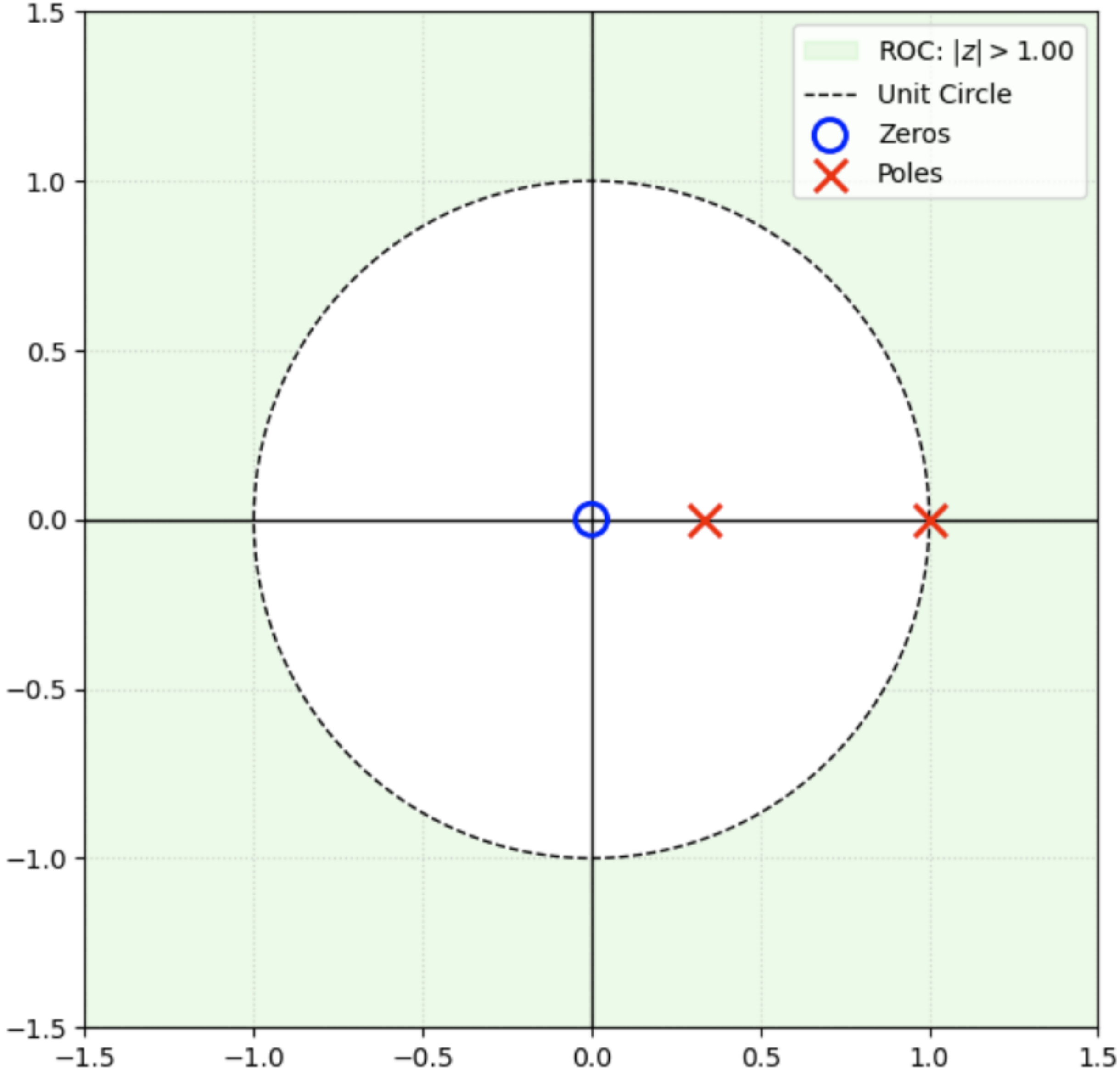
a = [3, -4, 1]

z, p, c = scipy.signal.tf2zpk(b,a)

plot_z_plane(z,p,show_roc=0)

plot_z_plane(z,p,show_roc=1)




ROC: |z| > 1.00
Unit Circle
Zeros
Poles
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5

Example 11

Determine the convergence region of the following transfer function:

$$H(z) = \frac{2z^4 + 16z^3 + 44z^2 + 56z + 32}{3z^4 + 3z^3 - 15z^2 + 18z - 12}$$

Solution

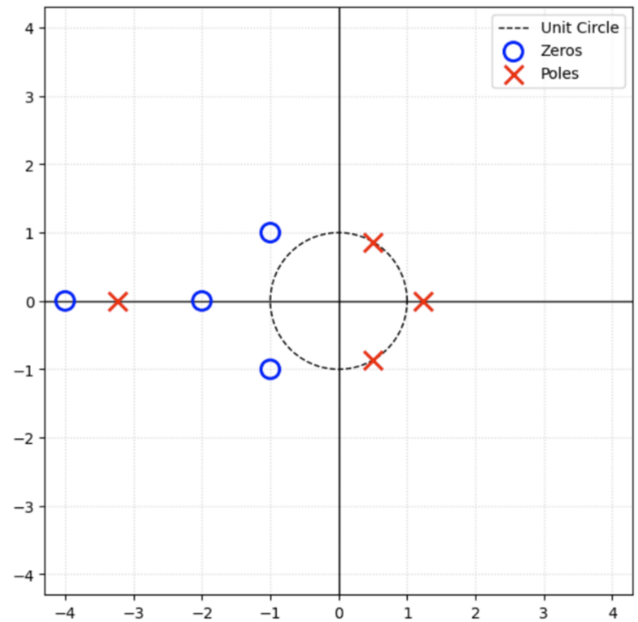
b = [2, 16, 44, 56, 32]

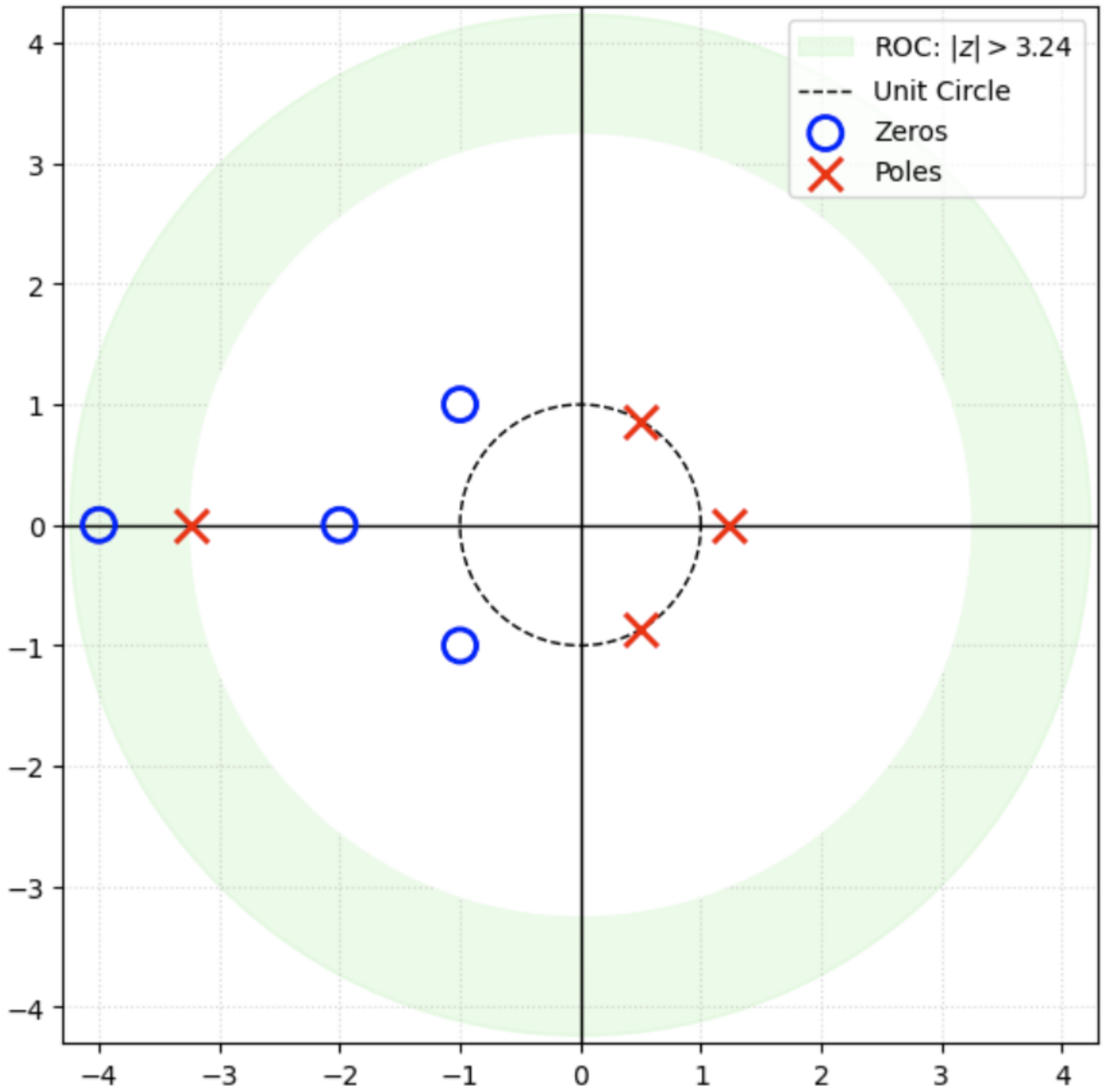
a = [3, 3, -15, 18, -12]

z, p, c = scipy.signal.tf2zpk(b,a)

plot_z_plane(z,p, show_roc=0)

plot_z_plane(z,p, show_roc=1)




ROC: |z| > 3.24
Unit Circle
Zeros
Poles
4
3
2
1
0
−1
−2
−3
−4
−4
−3
−2
−1
0
1
2
3
4