Digital Signal Processing Laboratory
Department of Electrical & Computer Engineering

Digital Signal Processing

Lab 6

Inverse Z-transform

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Dr. Konstantinos Karampidis

Introduction

With the inverse Z-transform we can compute the discrete-time signal x(n) when we know its Z-transform, X(z). To denote this operation, we write:

$$x(n) = Z^{-1}\{X(z)\}$$

Inverse Z-transform

A practical approach to compute the inverse Z-transform is to sum a few partial fractions and then use the table with the already computed Z-transforms of common sequences.

The function residuez

With residuez function we can find the appropriate coefficients to write the transfer function as a sum of partial fractions.

- ✓ The residuez function takes two arguments, the coefficients of the numerator (b) and denominator (a) in descending order of Z.
- ✓ It returns three coefficient matrices (r, p, k).

The function residuez

RESIDUEZ Z-transform partial-fraction expansion.
[R,P,K] = RESIDUEZ(B,A) finds the residues, poles and direct terms of the partial-fraction expansion of B(z)/A(z),

```
B(z)      r(1)                     r(n)
---- = -----------     +...  -----------      + k(1) + k(2)z^(-1) ...
A(z)   1-p(1)z^(-1)          1-p(n)z^(-1)
```

Example 1

Find the inverse Z-transform of the following transfer function:

$$G(z) = \frac{z}{3z^2 - 4z + 1}$$

Solution

```
b=[0 1 0];
a=[3 -4 1];
[r,p,k]=residuez(b,a)
```

The output should be

```
r = [0.5000  -0.5000]
p = [1.0000  0.3333]
k = [0]
```

Example 1

If we now replace the calculated r, p, k in the equation:

```
B(z)       r(1)                    r(n)
---- = ------------     +...  ------------      + k(1) + k(2)z^(-1) ...
A(z)   1-p(1)z^(-1)           1-p(n)z^(-1)
```

the transfer function can be written:

$$G(z)=\frac{z}{3z^2-4z+1}=0.5\frac{1}{1-z^{-1}}-0.5\frac{1}{1-0.333z^{-1}}$$

Example 1

$$G(z) = \frac{z}{3z^2 - 4z + 1} = 0.5\frac{1}{1 - z^{-1}} - 0.5\frac{1}{1 - 0.333z^{-1}}$$

Using the table of known Z-transforms we see that:

$$Z^{-1}\left\{\frac{1}{1 - z^{-1}}\right\} = 1^n u(n) \qquad \text{and} \qquad Z^{-1}\left\{\frac{1}{1 - 0.333z^{-1}}\right\} = 0.333^n u(n)$$

Therefore, the inverse Z-transform of G(z) is:

$$G(n) = 0.5u(n) - 0.5 \cdot 0.333^n u(n)$$

Example 2

Find the inverse Z-transform of the following transfer function:

$$H(z) = \frac{32z^2 + 4}{16z^2 + 12z + 2}$$

Solution

```
b=[32 0 4];
a=[16 12 2];
[r,p,k]=residuez(b,a)
```

The output should be

```
r = [6  -6]
p = [-0.5  -0.25]
k = [2]
```

Example 2

By using r, p and k we calculated the sum of partial fractions is:

$$H(z) = \frac{6}{1+0.5z^{-1}} - \frac{6}{1+0.25z^{-1}} + 2$$

By using the table of known Z-transforms we have:

$$Z^{-1}\{H(z)\} = H(n) = 6\cdot(-0.5)^n u(n) - 6\cdot(-0.25)^n u(n) + 2\delta(n)$$

Note: Suppose k=[2, 4, 1], then

$H(z) = \cdots + 2 + 4z^{-1} + z^{-2}$

$H(n) = \cdots + 2\delta(n) + 2\delta(n-1) + \delta(n-2)$

Example 3

Find the inverse Z-transform of the following transfer function:

$$H(z) = \frac{18z^3}{18z^3 + 3z^2 - 4z - 1}$$

Solution

```
b=[18 0 0 0];
a=[18 3 -4 -1];
[r,p,k]=residuez(b,a)
```

The output should be

r = [0.36 0.24 0.4]

p = [0.5 -0.33 -0.33]

k = [0]

Example 3

If we have more than one identical pole in p (as in our exercise where we have 2 times the value -0.333) then these terms should be written as shown below.

```
      R(j)                R(j+1)                          R(j+m-1)
--------------  + ------------------   + ... +  ------------------
1 - P(j)z^(-1)    (1 - P(j)z^(-1))^2            (1 - P(j)z^(-1))^m
```

where P(j) = ... = P(j+m-1) is a pole of plurality m.

Therefore :

$$H(z) = 0.36\frac{1}{1-0.5z^{-1}} + 0.24\frac{1}{1+0.33z^{-1}} + 0.4\frac{1}{\left(1+0.33z^{-1}\right)^{2}}$$

Example 3

$$H(z) = 0.36\frac{1}{1-0.5z^{-1}} + 0.24\frac{1}{1+0.33z^{-1}} + 0.4\frac{1}{\left(1+0.33z^{-1}\right)^2}$$

By using the table of the known Z-transforms we notice that

$$Z^{-1}\left\{\frac{1}{1-0.5z^{-1}}\right\} = 0.5^n u(n) \quad \text{και} \quad Z^{-1}\left\{\frac{1}{1+0.33z^{-1}}\right\} = -0.33^n u(n)$$

The fraction that remains is similar to the 5th Z-transform on the table and therefore:

$$\frac{1}{\left(1+0.33z^{-1}\right)^2} = \frac{1}{-0.33z^{-1}} \cdot \frac{-0.33z^{-1}}{\left(1+0.33z^{-1}\right)^2} = \frac{1}{-0.33} \cdot z \cdot \frac{-0.33z^{-1}}{\left(1+0.33z^{-1}\right)^2}$$

$$z^{-1}\left\{\frac{-0.33z^{-1}}{\left(1+0.33z^{-1}\right)^2}\right\} = n(-0.33)^n u(n)$$

We notice that a "z" remained to the previous equation

By using the shifting property $x(n-n_0) \overset{Z}{\leftrightarrow} z^{-n_0}X(z)$

$$\frac{1}{-0.33}\cdot z\cdot\frac{-0.33z^{-1}}{\left(1+0.33z^{-1}\right)^2} = \frac{1}{-0.33}\cdot z^{-(-1)}X(z)$$

We notice that n_o=-1. This means that the term $n(-0.33)^n u(n)$ will be $(n+1)(-0.33)^{n+1}u(n+1)$

Therefore, the transfer function is:

$$H(n) = 0.36\cdot 0.5^n u(n) + 0.24(-0.33)^n u(n) - \frac{0.4}{0.33}(n+1)(-0.33)^{n+1}u(n+1)$$

Example 4

Find the inverse Z-transform of the following transfer function:

$$X(z) = \frac{z^2}{z^3 - 4z^2 + 5z - 2}$$

Solution

```
b=[0,1,0,0];
a=[1,-4,5,-2];
[r, p, k]=residuez(b,a)
```

The output should be

```
r =
   2.0000
  -1.0000 - 0.0000i
  -1.0000 + 0.0000i

p =
   2.0000
   1.0000 + 0.0000i
   1.0000 - 0.0000i

k =
     0
```

Example 4

By using the calculated r, p, k values we get the following sum of partial fractions

$$X(z)=\frac{2}{1-2z^{-1}}-\frac{1}{1-z^{-1}}-\frac{1}{\left(1-z^{-1}\right)^{2}}=\frac{2}{1-2z^{-1}}-\frac{1}{1-z^{-1}}-\frac{1}{z^{-1}}\cdot\frac{z^{-1}}{\left(1-z^{-1}\right)^{2}}$$

$$=\frac{2}{1-2z^{-1}}-\frac{1}{1-z^{-1}}-z\cdot\frac{z^{-1}}{\left(1-z^{-1}\right)^{2}}$$

We then use the table of known Z-transforms

$$x(n)=2\cdot 2^{n}u(n)-u(n)-(n+1)\cdot u(n+1)$$