Discrete-Time Signal Processing

Lectures 18-19

FIR Digital Filter Design Methods

Designing FIR Filters

- Advantages of FIR filters:
  1. more phase control--can design filters that are exactly linear phase
  2. guaranteed stability--FIR filters have only zeros, no non-zero, finite poles
  3. can match any arbitrary design specification to arbitrary precision with sufficient filter length
  4. several excellent, well-understood, design techniques
  5. easy to implement

FIR Filter Basics

- FIR filters have the characteristic:

  $h(n) \neq 0$ for a finite range of n (e.g., $n = 0, 1, \ldots, M$)

- FIR filters designed to match ideal frequency response:

  $H_{id}(e^{j\omega}) \longleftrightarrow h_{id}(n)$ (which is infinite in extent)

- How to do this?

Windowing Method

$\Rightarrow$ use a Rectangular Window to weight the ideal response

Windowing Method

- Generalize this concept of weighting the ideal sequence by a finite duration window $\Rightarrow$ Window Design Method

Want $W(e^{j\omega})$ to be as close to an impulse as possible, in order to reduce the negative effects of ringing => need a window with what?

Window Designs

- Example--Rectangular Window

$$w(n) = 1 \qquad 0 \le n \le M$$

$$= 0 \qquad otherwise$$

$$W\left(e^{j\omega}\right) = e^{-j\omega M/2} \frac{\sin\left[\omega/2\left(M+1\right)\right]}{\sin(\omega/2)}$$

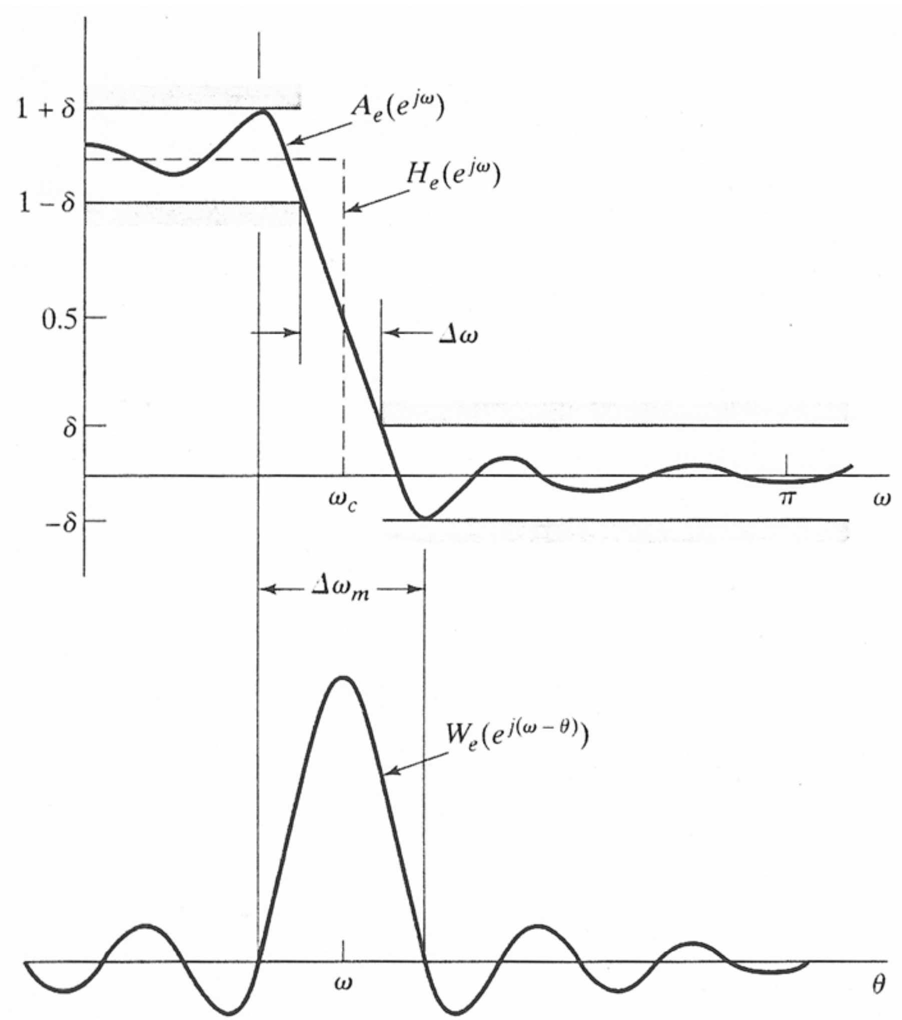
Window Designs

- Rectangular Window Properties:
  1. main lobe width controls:
  2. side lobe area controls:

- If M increases $\Rightarrow$

- The sharp transitions of the windowed sequence at $n = 0$ and $n = M$ cause large ripples in the filter $\Rightarrow$ use more gradually tapering windows
- As $M \to \infty$, the rectangular window corresponds to no windowing, i.e., $W(e^{j\omega}) = \delta(\omega)$ and $H(e^{j\omega}) = H_{id}(e^{j\omega})$ (i.e., no truncation of $h_{id}(n)$)

Effect of Window at Discontinuity

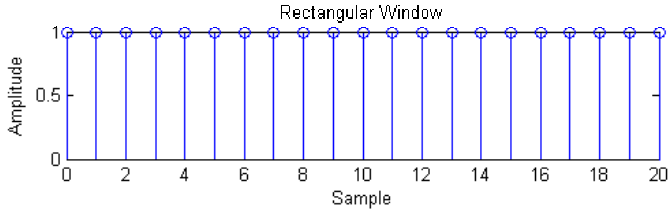


- Consider ideal LPF with cutoff ω_c
- The window frequency response is centered on the discontinuity
  - width between peak overshoots is window main lobe width
  - approximation is symmetric around $\omega = \omega_c$

Window Designs

1. Rectangular: $w_R(n) = 1 \quad 0 \le n \le M$

$= 0 \quad otherwise$

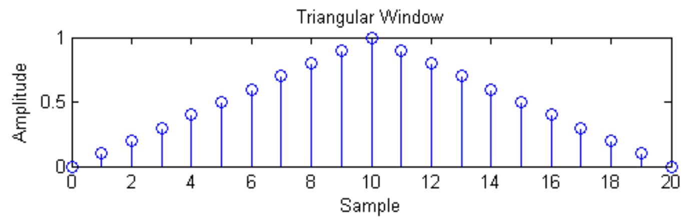


Main Lobe Width $\sim \dfrac{4\pi}{(M+1)}$

Peak Side Lobe Amplitude $= -13$ dB

2. Bartlett (Triangular):

$$w_B(n) = 2n/M \quad 0 \le n \le M/2$$
$$= 2 - 2n/M \quad M/2 < n \le M$$
$$= 0 \quad otherwise$$



Main Lobe Width $\sim \dfrac{8\pi}{M}$

Peak Side Lobe Amplitude $= -25$ dB

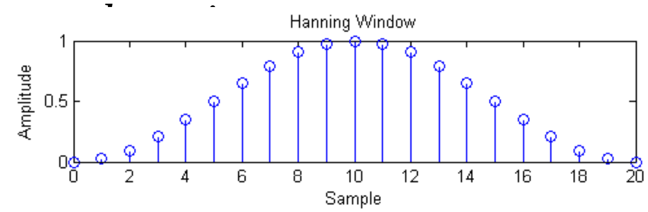
$w_B(n) = w_R(n) * w_R(n)$ $(M/2$ point Rect. window)

$\Rightarrow$ Main Lobe twice as large as for RW

Window Designs

3. Hanning:

$$w_{HN}(n) = 0.5 - 0.5\cos(2\pi n / M) \qquad 0 \le n \le M$$
$$= 0$$

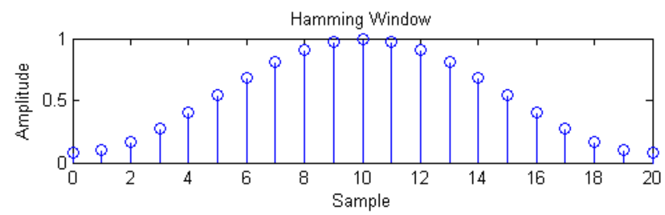


Main Lobe Width $\sim \dfrac{8\pi}{M}$

Peak Side Lobe Amplitude $= -31$ dB

4. Hamming:

$$w_{HM}(n) = 0.54 - 0.46\cos(2\pi / M) \qquad 0 \le n \le M$$
$$= 0 \qquad \textit{otherwise}$$



Main Lobe Width $\sim \dfrac{8\pi}{M}$

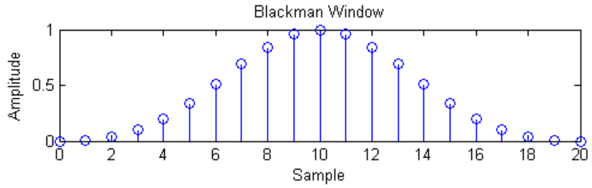
Peak Side Lobe Amplitude $= -41$ dB

Window Designs

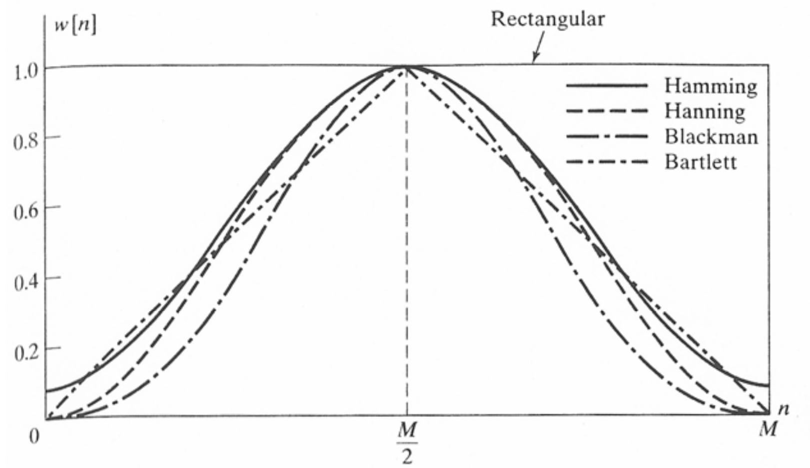
5. Blackman:

$$w_B(n) = 0.42 - 0.5\cos(2\pi n/M) + 0.08\cos(4\pi n/M) \qquad 0 \le n \le M$$
$$= 0 \qquad otherwise$$

Main Lobe Width $\sim \dfrac{12\pi}{M}$

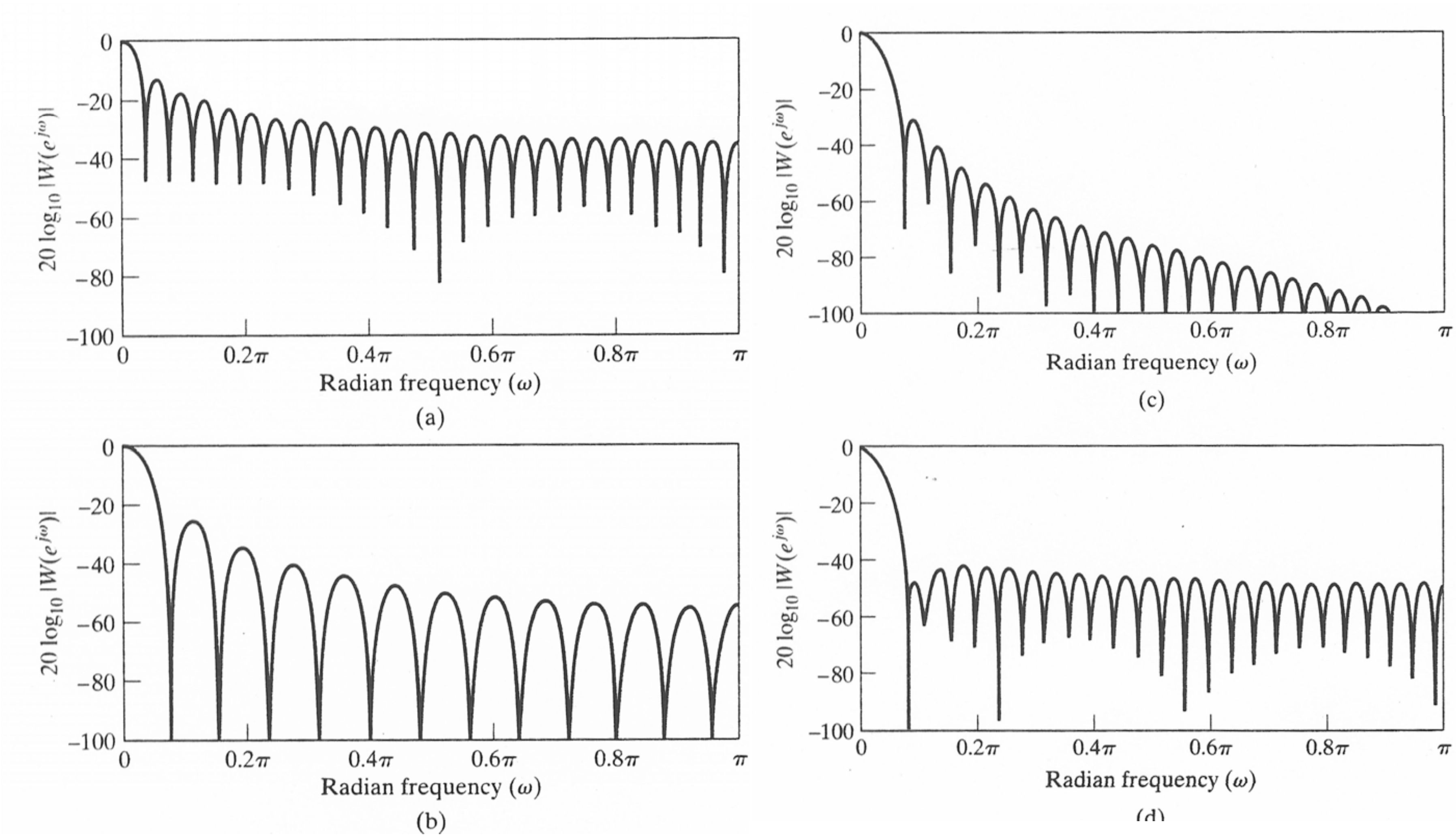


Peak Side Lobe Amplitude $= -57$ dB



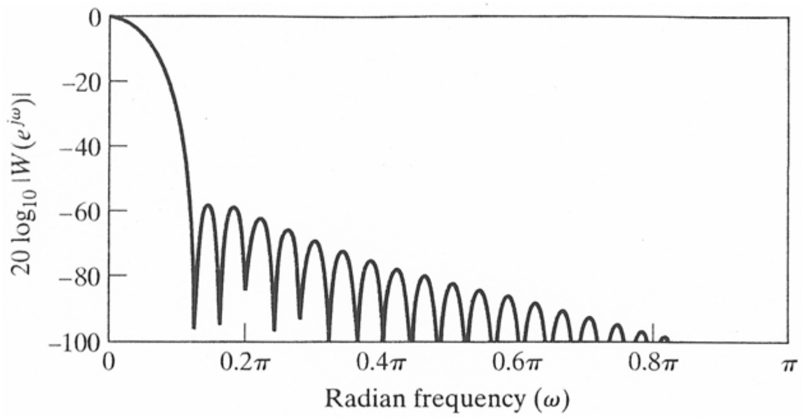
Note: all windows are symmetric =>

Window Frequency Responses



Log magnitude responses of windows with M=50; a) Rectangular, b) Triangular, c) Hanning, d) Hamming

Window Frequency Responses



(e)

Log magnitude response for M=50 Blackman window

TABLE 7.1 COMPARISON OF COMMONLY USED WINDOWS

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20 \log_{10} \delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hanning	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

Window Designs

- All windows have the following general properties:
  - their frequency responses are concentrated around $\omega = 0$
  - it is easy to compute the window, $w(n)$
  - we can write the window frequency responses as sums of shifted versions of the frequency response of the Rectangular window
- Window Comparisons (from table on previous slide)

Window Filter Design Example

Linear Phase Designs

- All the windows we have talked about are symmetric around sample $M/2$ for $M+1$ point windows; therefore

  $$W(e^{j\omega}) = W_e(e^{j\omega})\, e^{-j\omega M/2}$$

- If the ideal filter response is also symmetric around sample $M/2$ for an $M+1$ point filter duration, then

  $$H_{id}(e^{j\omega}) = H_e(e^{j\omega})e^{-j\omega M/2}$$

- Then we have the result:

Kaiser Window Designs

- The real window design problem is finding a window that meets specifications on both transition width and ripple with any degree of 'optimality' $\Rightarrow$ the Kaiser window meets this need
- Kaiser Window Design Problem: find a function that is maximally concentrated around $\omega = 0$; the near-optimal solution is found using Bessel functions (these are related to the Prolate Spheroidal Wave Functions which are the minimum time-space product kernals)

$$w(n) = \frac{I_0\left[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}\right]}{I_0(\beta)} \qquad 0 \le n \le M$$

$$= 0 \qquad otherwise$$

with $\alpha = M/2$, and $I_0(\cdot)$ being the zeroth order modified Bessel function of the first kind

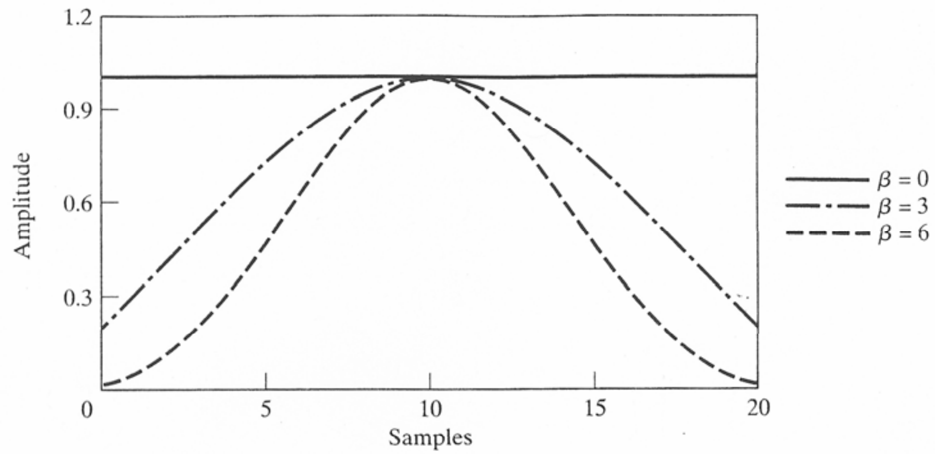
- Window parameters: β, M
- Main lobe width and side lobe height can be traded off against each other by varying these parameters

Kaiser Window Designs

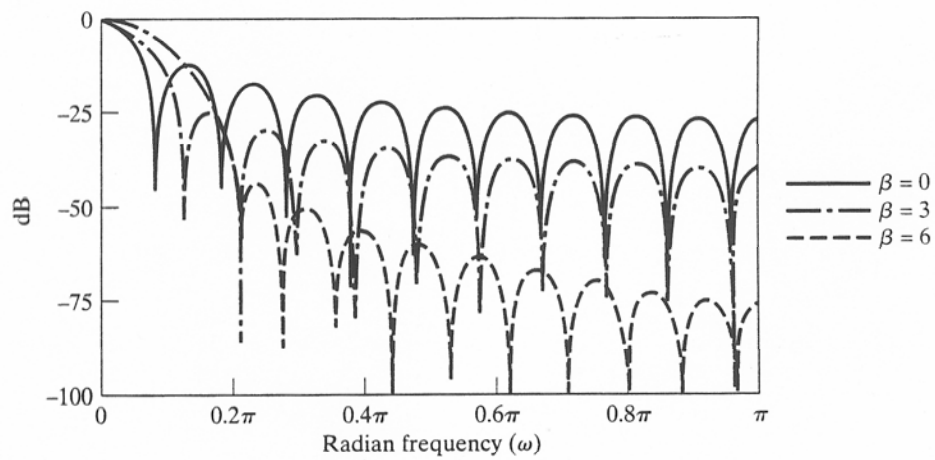
- We can determine β, M exactly based on desired ripple, δ, and transition band width, $\Delta\omega = \omega_s - \omega_p$
- Let $A = -20\log 10(\delta)$

$\therefore$ can find Kaiser window parameters easily based on desired filter specifications $\Rightarrow$ removes the trial-and-error approach and iterations for finding the best/most appropriate window

Kaiser Window Designs



(a)



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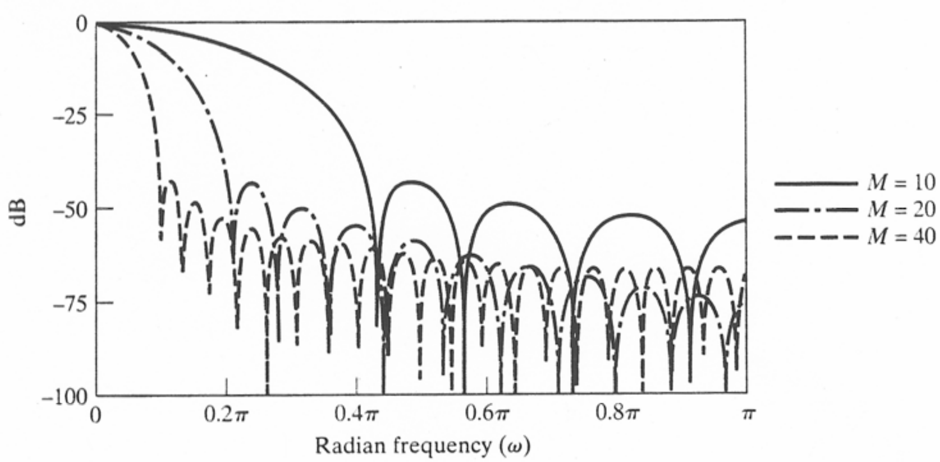


(c)

a) Kaiser windows for β=0, 3, and 6 and M=20; b) Fourier transforms corresponding to windows in a); c) Fourier transforms of Kaiser windows with β=6 and M=10,20 and 40

Kaiser Window LPF

- LPF Specifications:

  $$\omega_p = 0.4\pi;\ \omega_s = 0.6\pi,\ \delta_1 = 0.01,\ \delta_2 = 0.001$$

- Since window designed filters must have the same ripple specs, we set $\delta = 0.001$
- Set ideal cutoff frequency to be:

  $$\omega_c = \frac{\omega_p + \omega_s}{2} = 0.5\pi;\ \Delta\omega = \omega_s - \omega_p = 0.2\pi$$

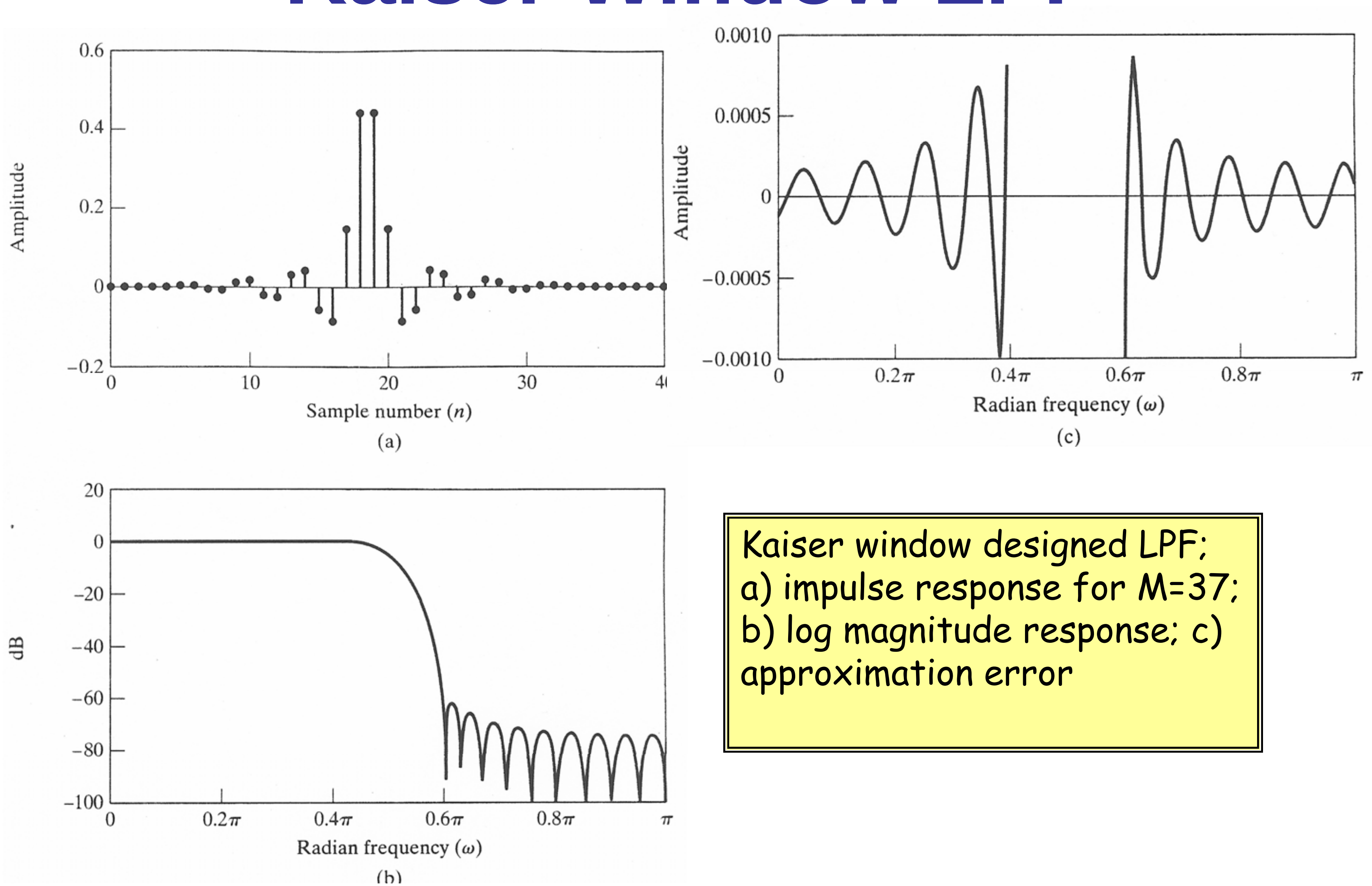
- Kaiser window parameters:

  $$A = -20\log 10(\delta) = 60 \Rightarrow \beta = 5.653,\ M = 37$$

- Determine impulse response as:

  $$h(n) = \frac{\sin\left[\omega_c(n-\alpha)\right]}{\pi n-\alpha)} \cdot \frac{I_0\left[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}\right]}{I_0(\beta)} \quad 0 \le n \le M$$

Kaiser Window LPF



Kaiser window designed LPF;
a) impulse response for M=37;
b) log magnitude response; c)
approximation error

Equiripple Filter Design

- The windowing method enabled us to design digital filters that were minimum mean squared error designs:

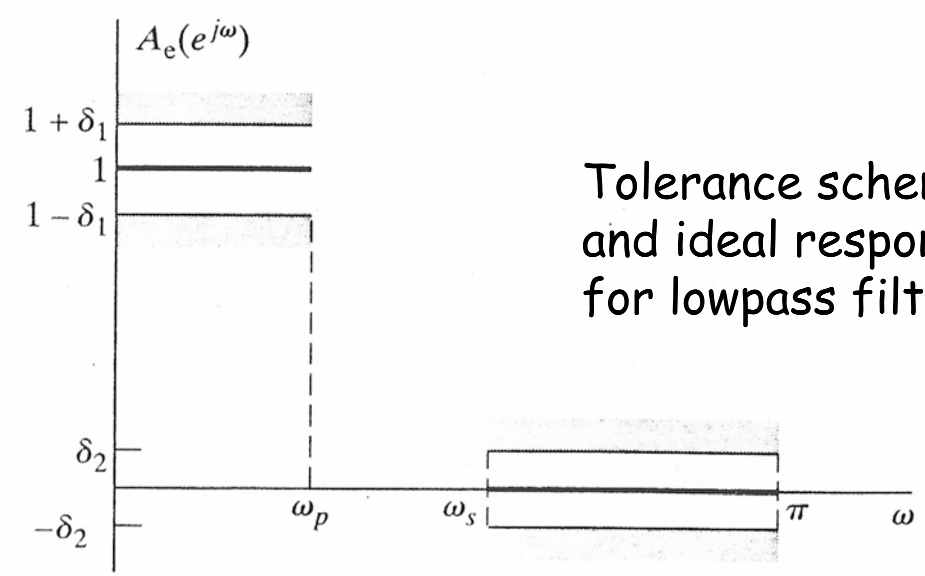
$$\min_{h(n)} \int_{-\pi}^{\pi} \left|E(\omega)\right|^2 d\omega = \min_{h(n)} \int_{-\pi}^{\pi} \left|H_{id}(e^{j\omega}) - H(e^{j\omega})\right|^2 d\omega$$

- There are other criteria that can be used to design digital filters including:

  - minimax ripple: $\min_{h(n)} \left[\max_{\omega} E(\omega) \right]$

- Such designs are called equiripple designs

Equiripple Filter Design



Tolerance scheme and ideal response for lowpass filter

- Given the ideal response, $H_{id}(e^{j\omega})$ in only the passband and stopband, find $h(n),\ n = 0,1,\ldots,M$ with (generalized) linear phase such that it achieves the minimum values of δ_1 or δ_2 or both simultaneously
- Design issues:
  1. recognizing when the given filter is optimum $\Rightarrow$ alternation theorem
  2. determining the coefficients of the optimal filter $\Rightarrow$ remez algorithm

Equiripple Filter FR

- Assume $h(n)$ has even symmetry, i.e., $h(n) = h(M-n)$ and odd length (M is even) $\Rightarrow$ Type I filter

$$h_e(n) = h(n + M/2) \Rightarrow h_e(n) = h_e(-n), \text{ zero phase FIR}$$

$$H(e^{j\omega}) = e^{-j\omega M/2} A_e(e^{j\omega})$$

Equiripple Filter FR

- We can express the function $\cos(\omega n)$ as a Chebyshev polynomial in $\cos(\omega)$, i.e.,

- We can now express $A_e(e^{j\omega})$ as:

Equiripple Filter FR

Alternation Theorem

- Note that:

- Thus we see that derivatives of $A_e(e^{j\omega})$ are zero where the derivatives of $P(x)$ are zero and at $\omega = 0, \pi$
- We can thus define the Alternation Theorem as a weighted approximation error function of the type:

Alternation Theorem

Minimum error achieved when all error are equiripple

Alternation Theorem

- Recall the problem we are trying to solve, namely we want to find $A_e\left(e^{j\omega}\right)$ to minimize the maximum error, i.e.,

$$\min_{\{h_e(n):0\le n\le L\}}\left(\max_{\omega\in F}\left|E(\omega)\right|\right)$$

where F defines the region of interest (the passband and the stop bands) and $E(\omega)$ is the weighted error function

- To get the solution we need to use the Alternation Theorem on the transformed problem

Alternation Theorem

- Let F_P denote the closed subset consisting of the disjoint union of closed subsets of the real axis x. $P(x)$ denotes an r^{th}-order polynomial

  $$P(x) = \sum_{k=0}^{r} a_k x^k$$

  Also, $D_P(x)$ denotes a given desired function of x that is continuous on F_P; $W_P(x)$ is a positive function, continuous on F_P, and $E_P(x)$ denotes the weighted error

  $$E_P(x) = W_P(x)\left[D_P(x) - P(x)\right].$$

  The maximum error $\|E\|$ is defined as

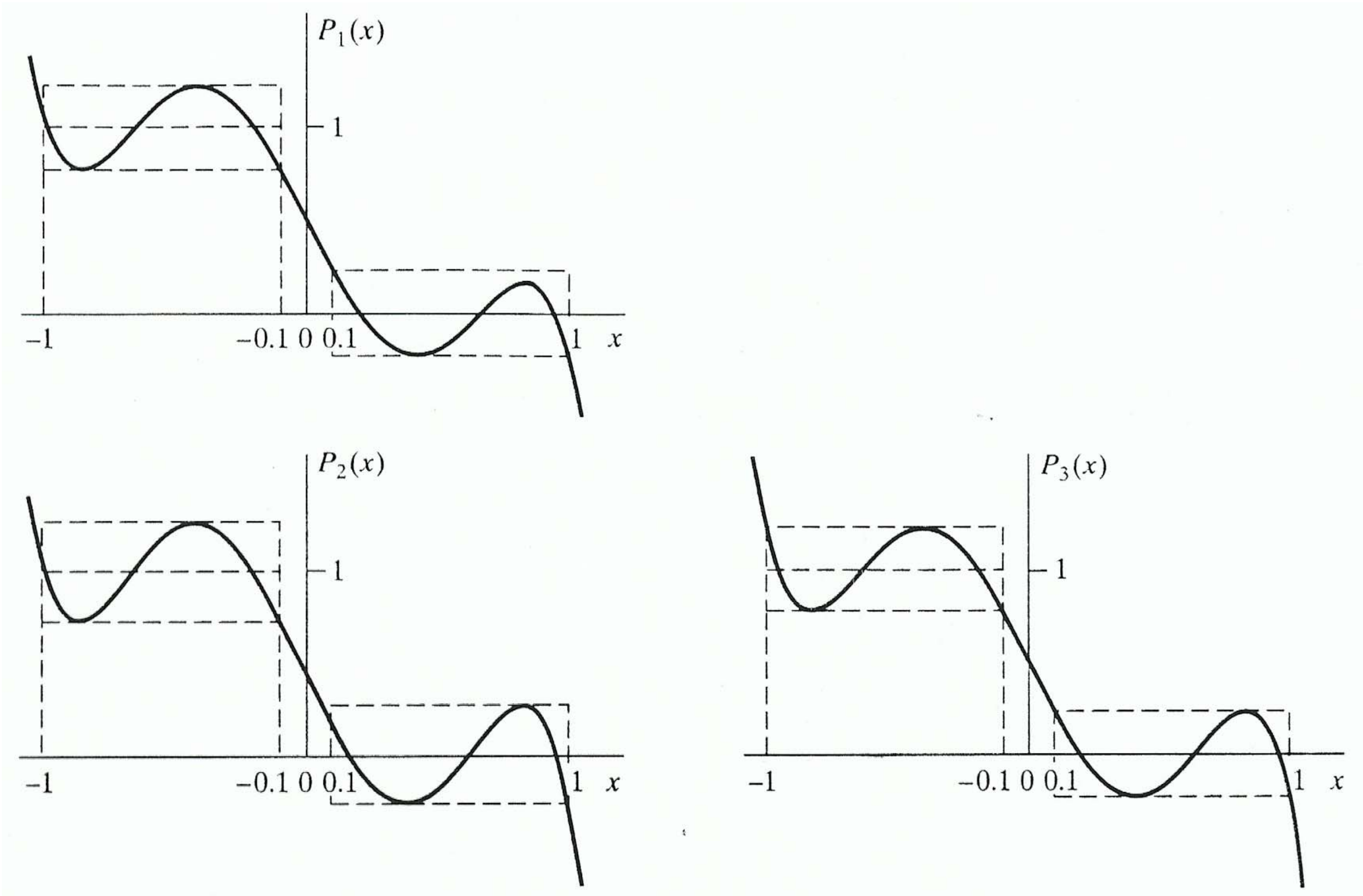
  $$\|E\| = \max_{x \in F_P} |E_P(x)|.$$

  A necessary and sufficient condition that $P(x)$ is the unique r^{th}-order polynomial that minimizes $\|E\|$ is that $E_P(x)$ exhibit *at least* $(r+2)$ alternations, i.e., there must exist at least $(r+2)$ values x_i in F_P such that $x_1 < x_2 < \ldots < x_{r+2}$ and such that $E_P(x_i) = -E_P(x_{i+1}) = \pm\|E\|$ for $i = 1, 2, \ldots, (r+1)$.

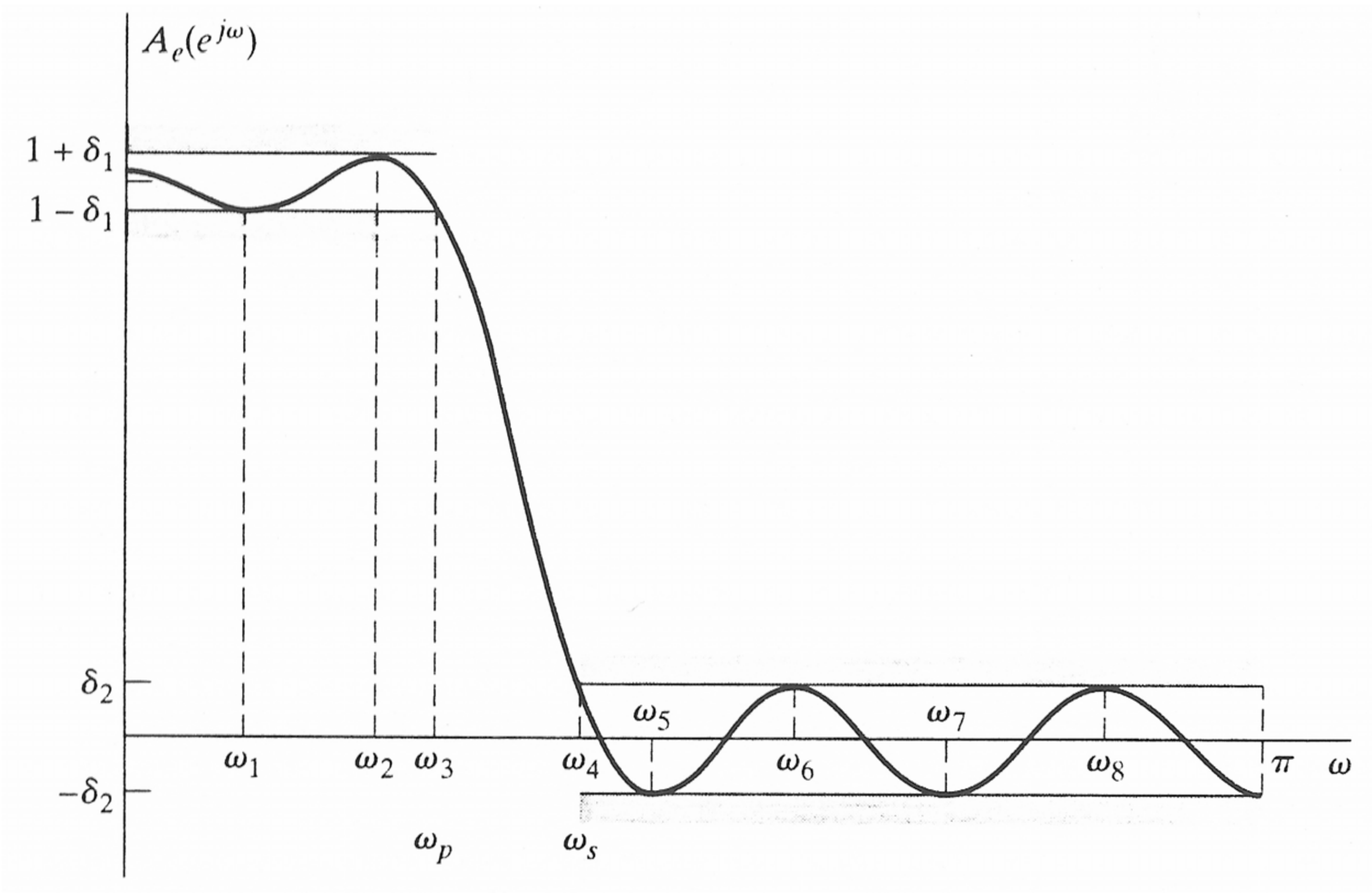
Alternation Theorem

- Alternations are points of maximum error and of alternating sign
- We use the alternation theorem to determine if the polynomial $P(x)$ is optimal $\Rightarrow$ filter design is optimal

Alternation Theorem



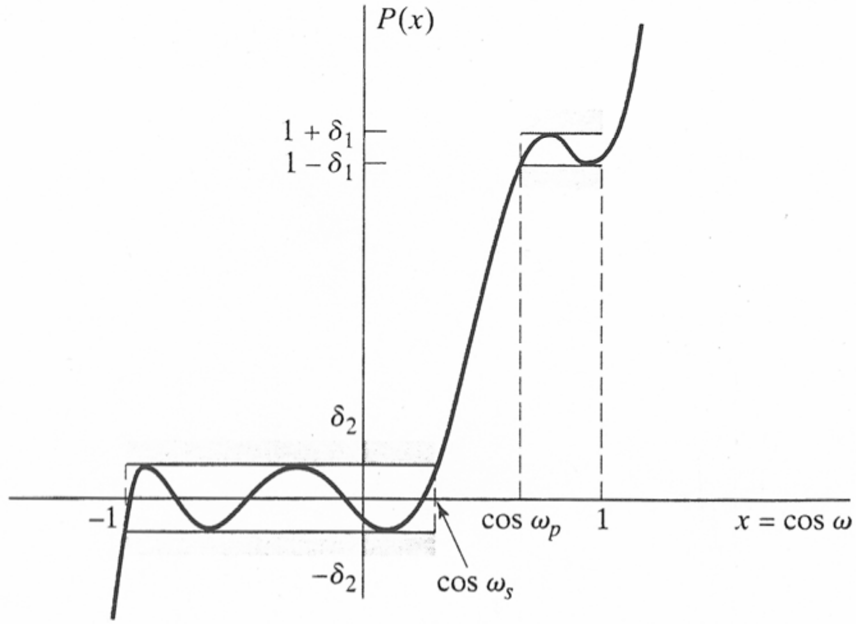
Alternation Theorem



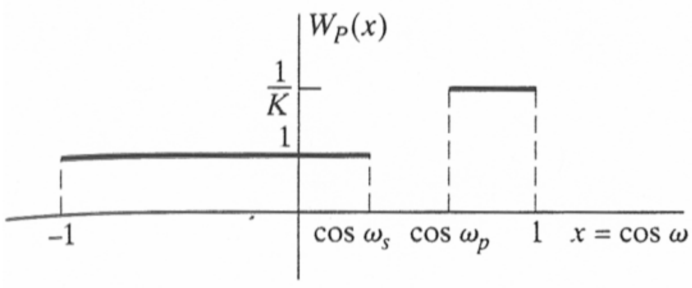
Alternation Theorem

Equivalent polynomial approximation functions as a function of x=cos(ω); a) approximating polynomial; b) weighting function; c) approximation error;

$K=\delta_1/\delta_2$



(a)



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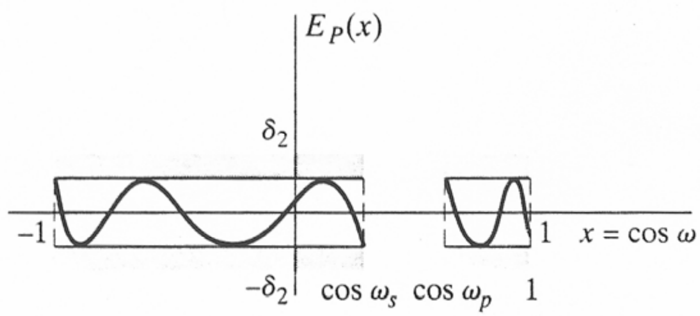
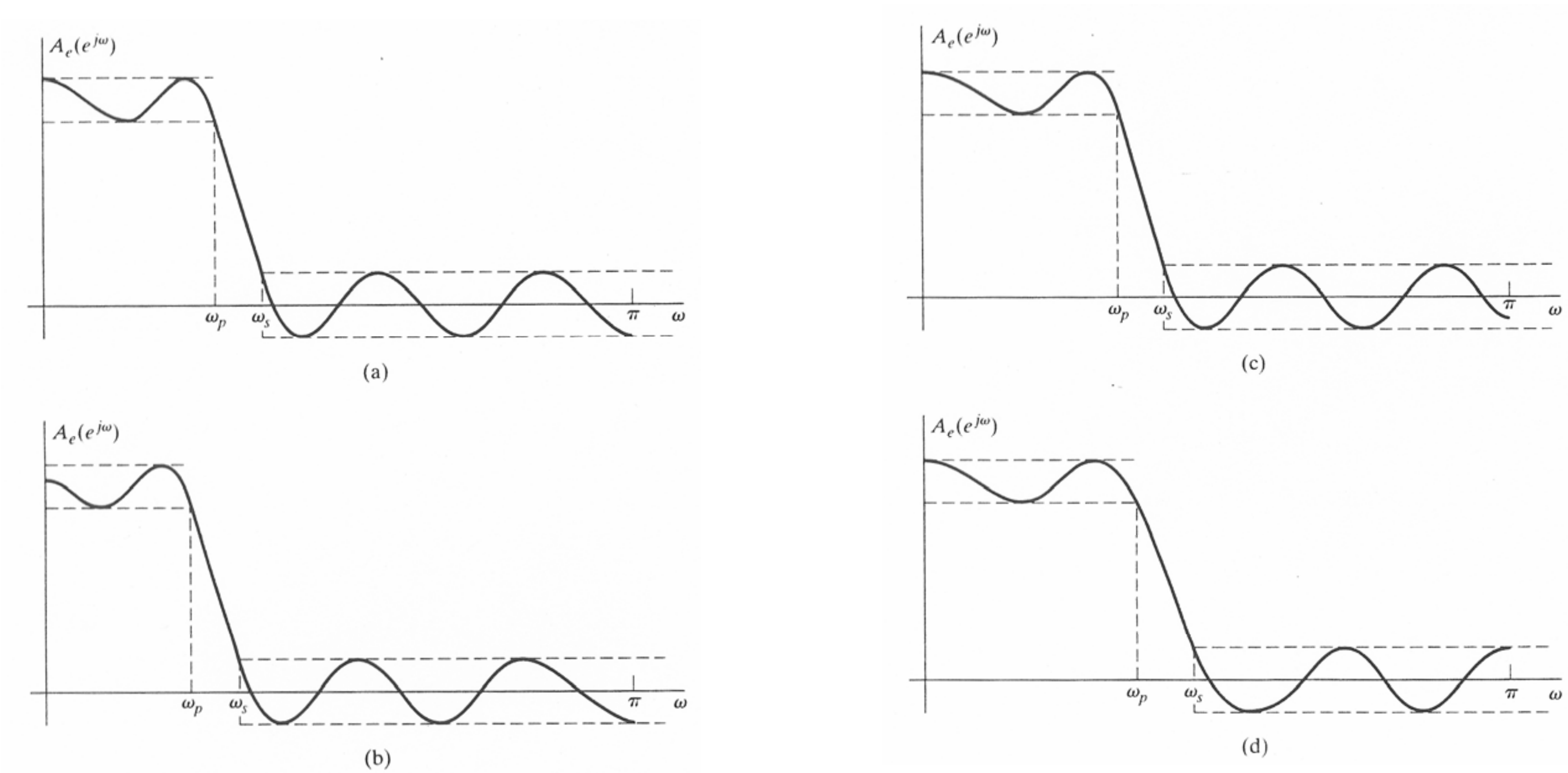


(c)

Alternation Theorem



Possible optimum lowpass filter approximations for L=7; a) L+3 alternations (extraripple case); b) L+2 alternations (extremum at ω=π); c) L+2 alternations (extremum at ω=0); d) L+2 alternations (extremum at both ω=0 and ω=π)

Alternation Theorem

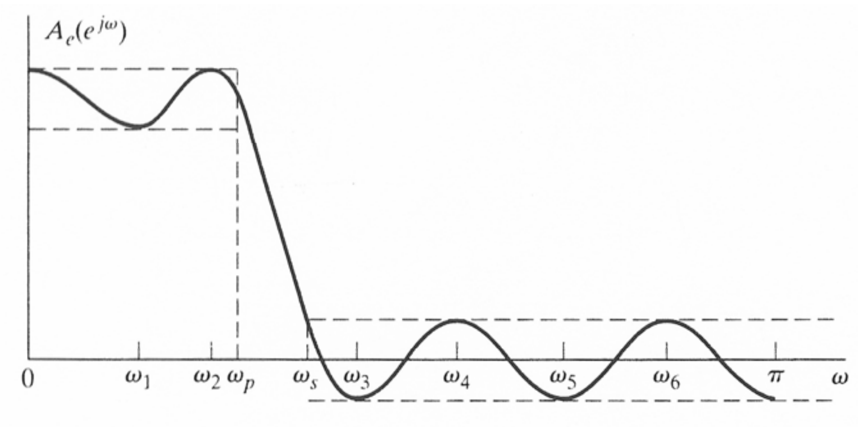


Illustration that the passband edge must be an alternation frequency

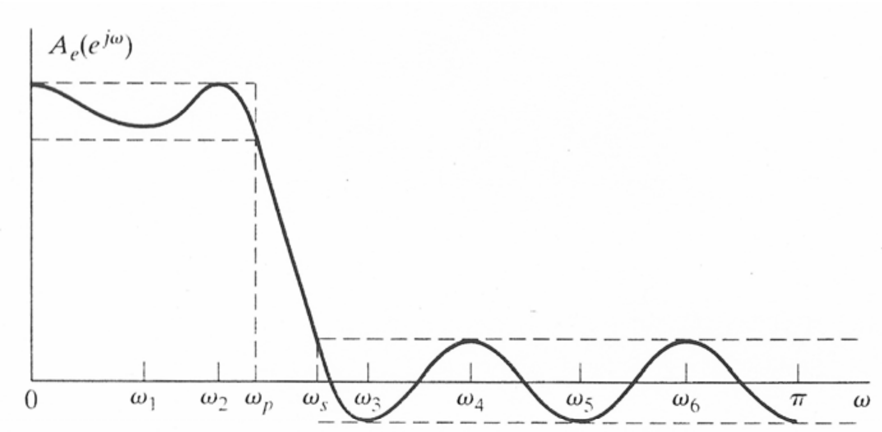


Illustration that the frequency response must be equiripple in the approximation bands

Alternation Theorem Examples

Alternation Theorem Examples

Optimal LPF Conditions

- Optimal (in a minimax sense) LPF satisfies the following conditions:
  1. minimum number of alternations $= L + 2$
  2. maximum number of alternations $= L + 3$ (extraripple case)
     --in the extraripple case $\omega = 0$ and $\omega = \pi$ are points of alternation
- To get extraripple designs must have alternations at:
  1. each of the 4 band edges $(\omega = 0, \omega_p, \omega_s, \pi)$
  2. internally at $L - 1$ points of zero slopes
  3. total of $L + 3$ alternations

Optimal LPF Conditions

- Lowpass Filter Conditions for Optimality:

1. Maximum number of alternations is $L+3$
2. Both ω_p and ω_s must be points of alternation
3. Optimum Type I filter is equiripple $\Rightarrow$ all points of zero slope inside the passband and stopband must be points of maximum error (except possibly at $0,\pi$)
4. Transition region must have monotone response

LPF Examples

Alternation Theorem Issues

- The Alternation Theorem says that there are $L+2$ alternation points for any type of filter
  - for lowpass/highpass filters, there are 4 band edges, $L-1$ zero slopes within bands for a total of $L+3$ maximum number of alternations
  - for bandpass filters, there are 6 band edges, $L-1$ zero slopes within bands for a total of $L+5$ maximum number of alternations; thus there is no problem with "lose one-lose two" alternations; also you can either lose one band edge as a point of alternation, or have one non-equiripple point and possible no alternation at $\omega=0,\pi$ and still be optimal

Alternation Theorem Examples

Parks-McClellan Filter Design

- Method for determining coefficients of optimal filters
  - Remez exchange algorithm used in the Parks-McClellan algorithm
- The optimal filter satisfies the relations:

$$W(\omega_i)\left[H_d(e^{j\omega_i}) - A_e\left(e^{j\omega_i}\right)\right] = (-1)^{i+1}\delta \qquad i = 1, 2, \ldots, (L+2)$$

$$\begin{bmatrix} 1 & x_1 & x_1^2 & & x_1^L & 1/W(\omega_1) \\ 1 & x_2 & x_2^2 & & x_2^L & -1/W(\omega_2) \\ . & . & . & . & . & . \\ 1 & x_{L+2} & x_{L+2}^2 & & x_{L+2}^L & (-1)^{L+2}/W(\omega_{L+2}) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ . \\ \delta \end{bmatrix} = \begin{bmatrix} H_d(e^{j\omega_1}) \\ H_d(e^{j\omega_2}) \\ \\ H_d(e^{j\omega_{L+2}}) \end{bmatrix}$$

$x_i = \cos(\omega_i)$, ω_i are points of alternations

Remez Exchange Algorithm

- Can use the above set of equations to solve for $a_i \rightarrow A_e(e^{j\omega})$
- The steps in the solution are as follows:

1. guess a set of $\{\omega_i\},\ i = 1, 2, \ldots, (L+2)$

   - ω_p and ω_s are fixed and must be 2 of the set of $\{\omega_i\}$, namely $\omega_l = \omega_p$ and $\omega_{l+1} = \omega_s$

2. can solve equations for a_i and δ using initial guess (Parks-McClellan algorithm finds δ and interpolates through $(L+1)$ points $(\omega_i, \pm\delta)$ to get $A_e(e^{j\omega})$)

3. if $|E(\omega)| \leq \delta\ \forall \omega \in$ the passband and stopband $\Rightarrow$ optimal filter found. Otherwise, find a new set of extremal frequencies using the $(L+2)$ largest peaks of the error in the current $A_e(e^{j\omega})$ (This is what is known as the Remez Exchange Algorithm)

Parks-McClellan / Remez Exchange Example

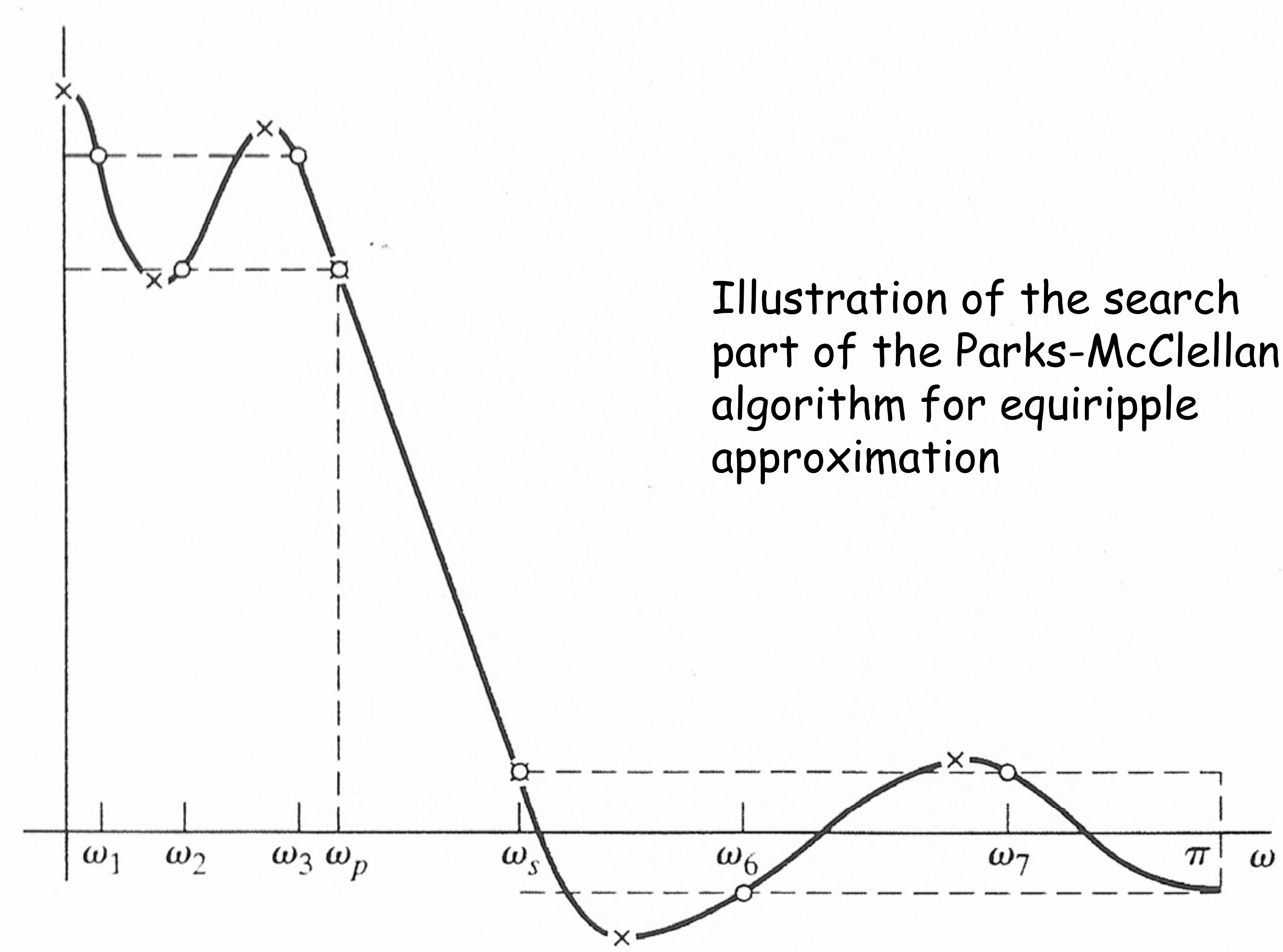
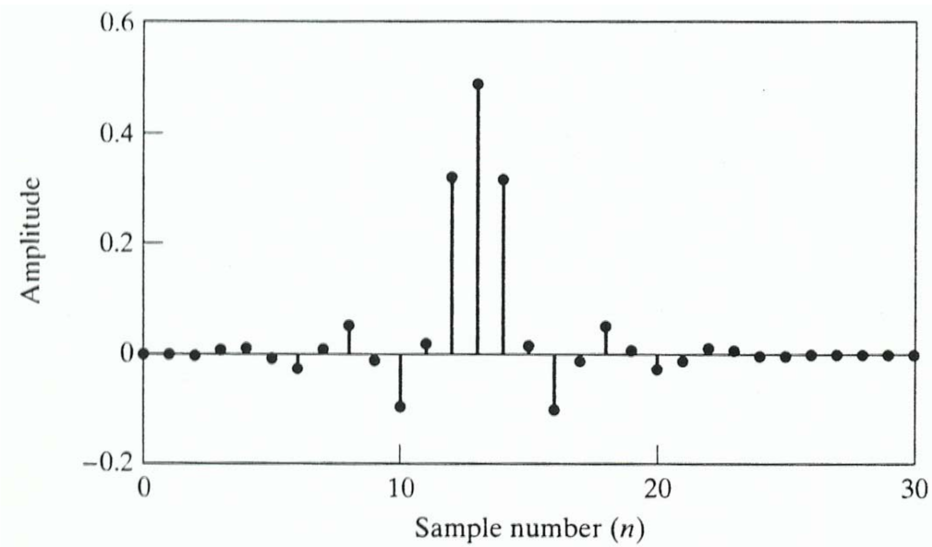


Illustration of the search part of the Parks-McClellan algorithm for equiripple approximation

Optimal FIR Filters



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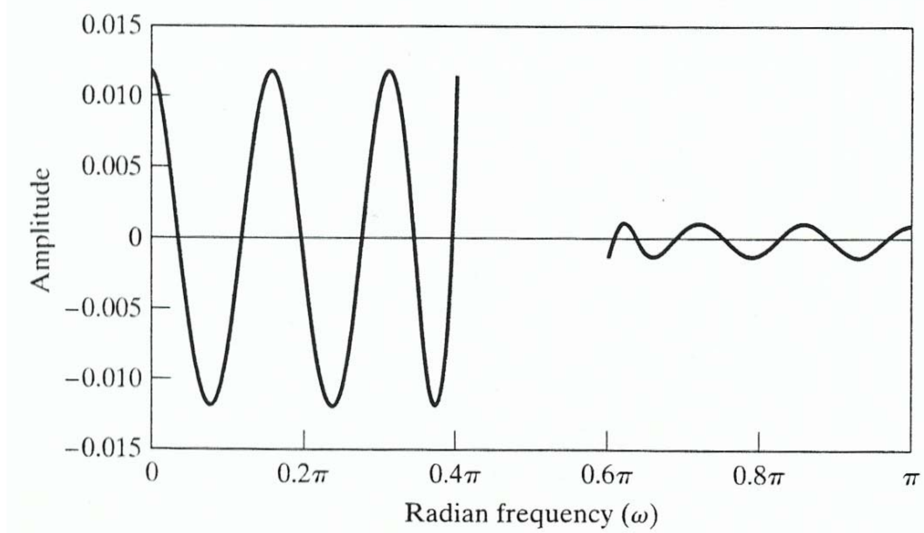

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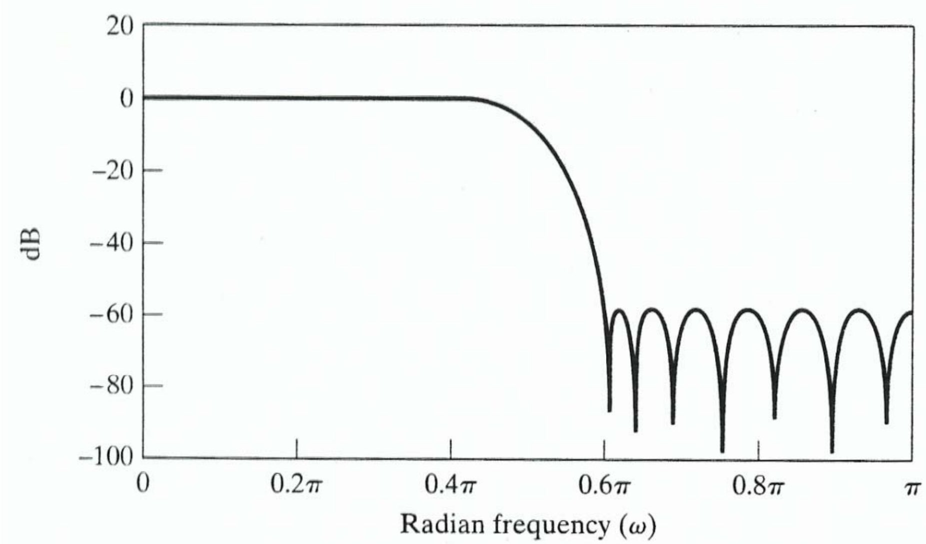
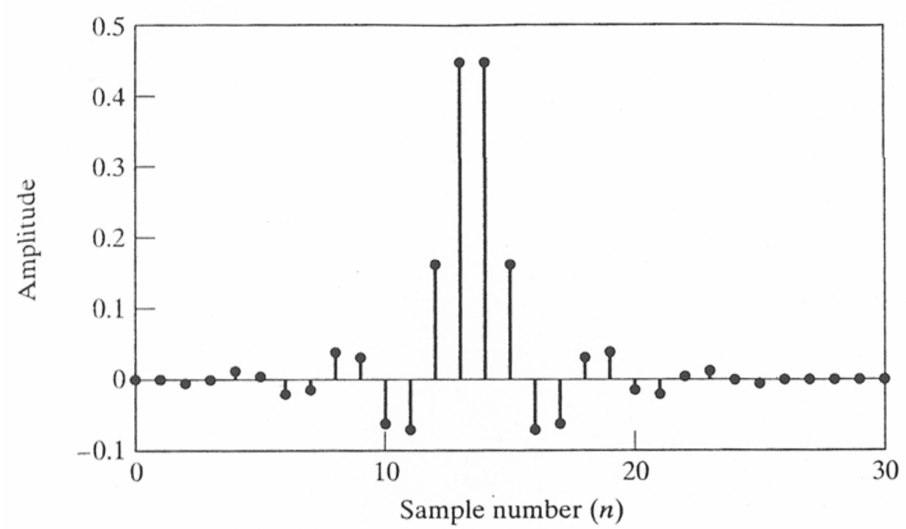


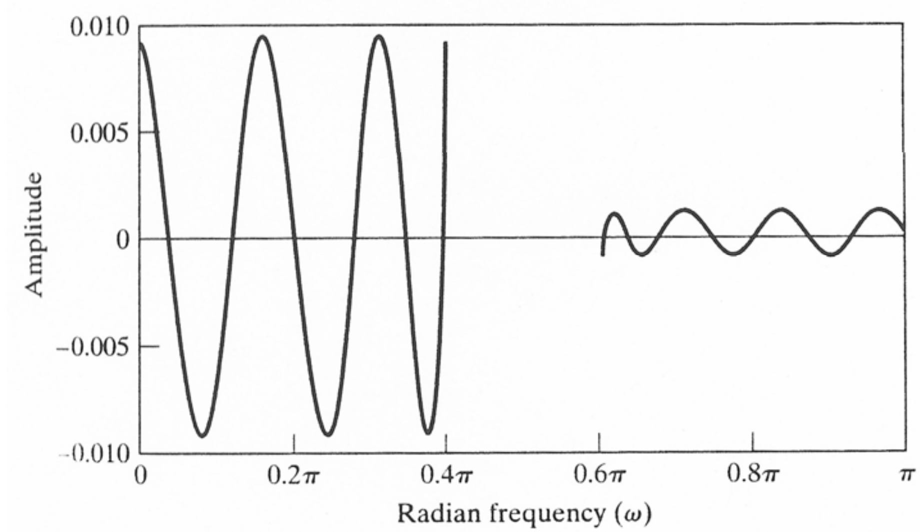
(b)

Optimum Type I FIR lowpass filter for $\omega_p=0.4\pi$, $\omega_s=0.6\pi$, $K=\delta_1/\delta_2$, and $M=26$; a) impulse response; b) log magnitude response; c) approximation error (unweighted)

Optimal FIR Filters



(a)



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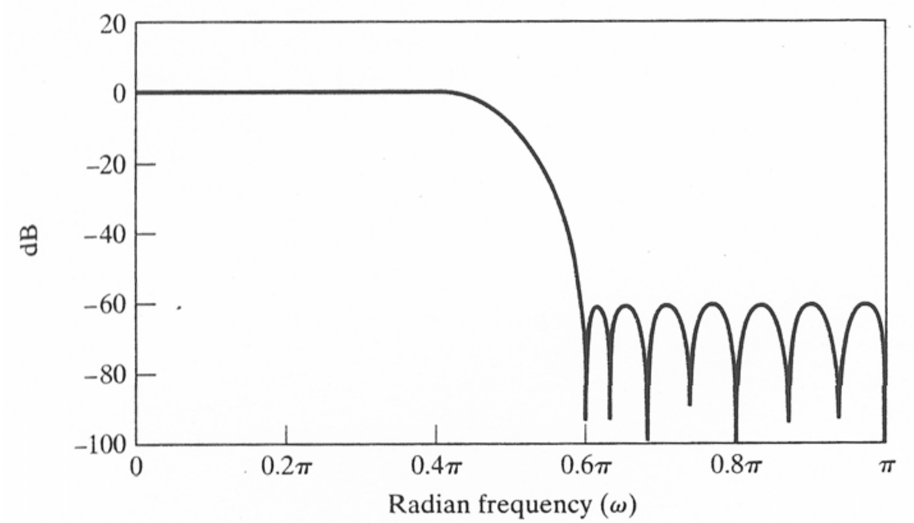

(b)

Optimum Type II FIR lowpass filter for $\omega_p=0.4\pi$, $\omega_s=0.6\pi$, $K=\delta_1/\delta_2$, and $M=26$; a) impulse response; b) log magnitude response; c) approximation error (unweighted)

Notice error at $\omega=\pi$

Remez Exchange Algorithm