Pre-Calculus Course

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Pre – Calculus Course

***The main objective** of this module is to prepare better the graduate students from high School to follow a University level module in Calculus and Linear Algebra*

The module apart of these lecture notes is also accompanied by: (a) Self – Evaluation Questions; (b) lecture notes; and (c) simulations/visualizations using GeoGebra (www.geogebra.org)

Chapter Five: Differentiation

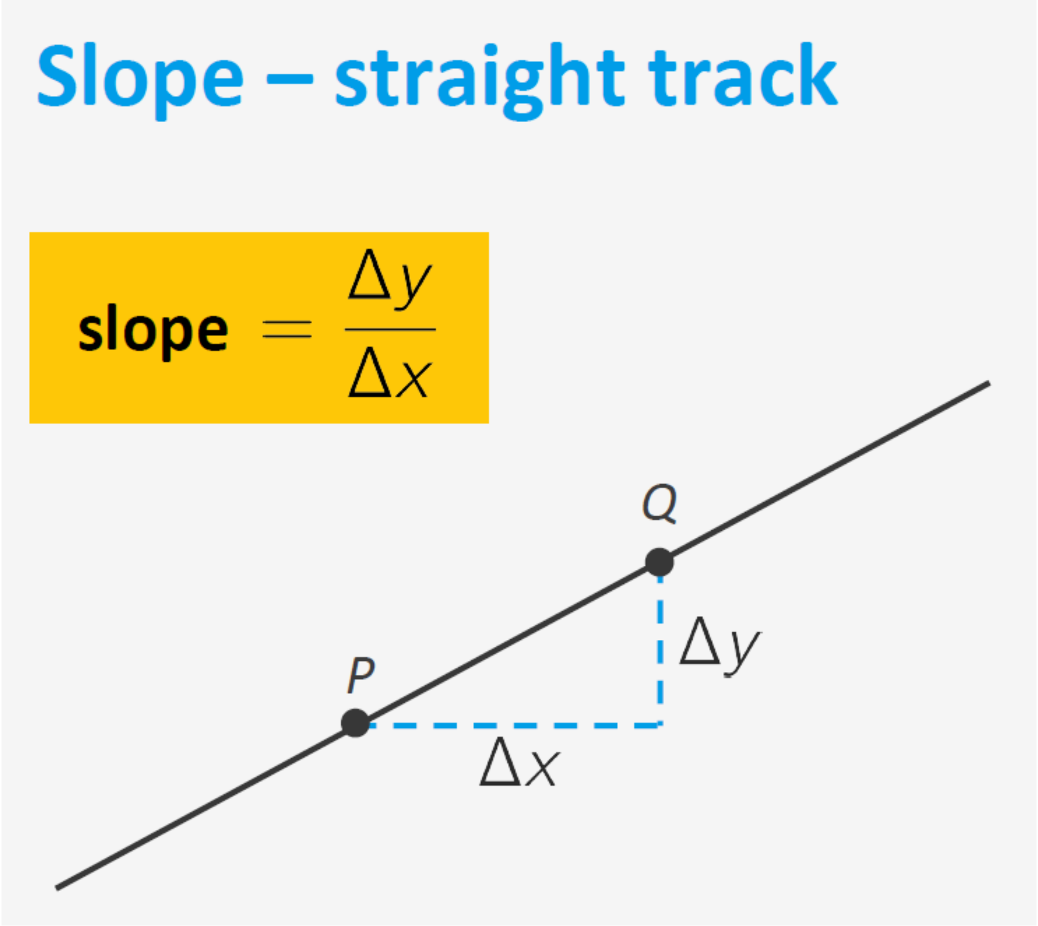
An Introduction

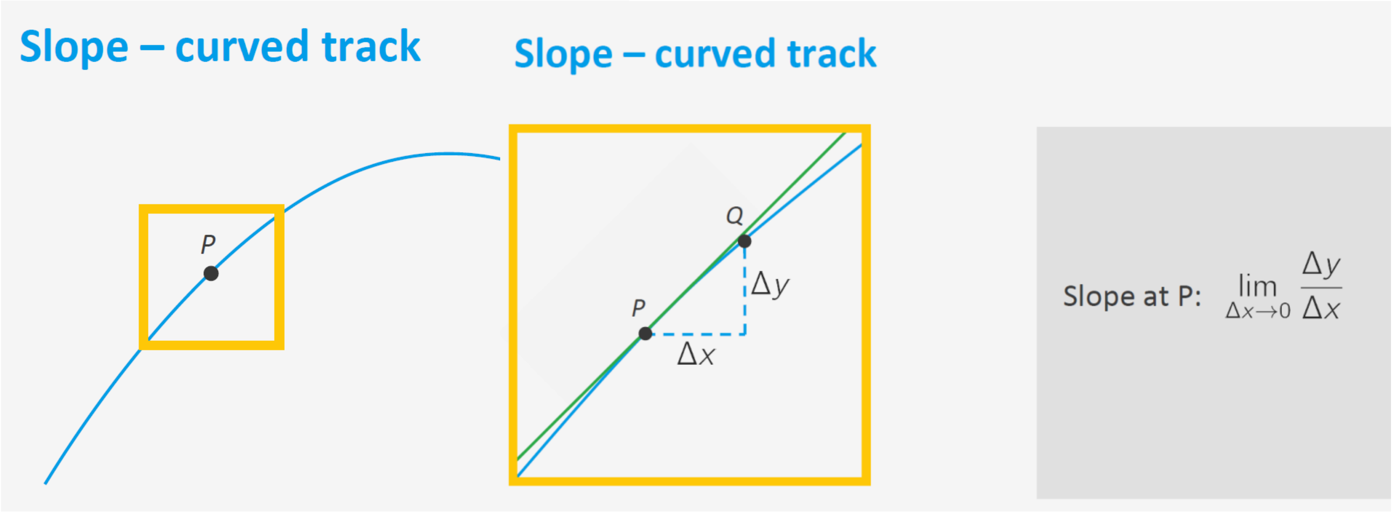
The aim of this chapter is: In this chapter we will learn **the definition of a derivative**. This can be used to calculate speed and to obtain the slope of a tangent line for instance. Then we will **discuss standard derivatives and rules of calculation**, followed by exercises to give you the chance to apply these rules. We will also discuss the concept of differentiability and the use of differentiation in optimization problems.

Definition of the Differentiation

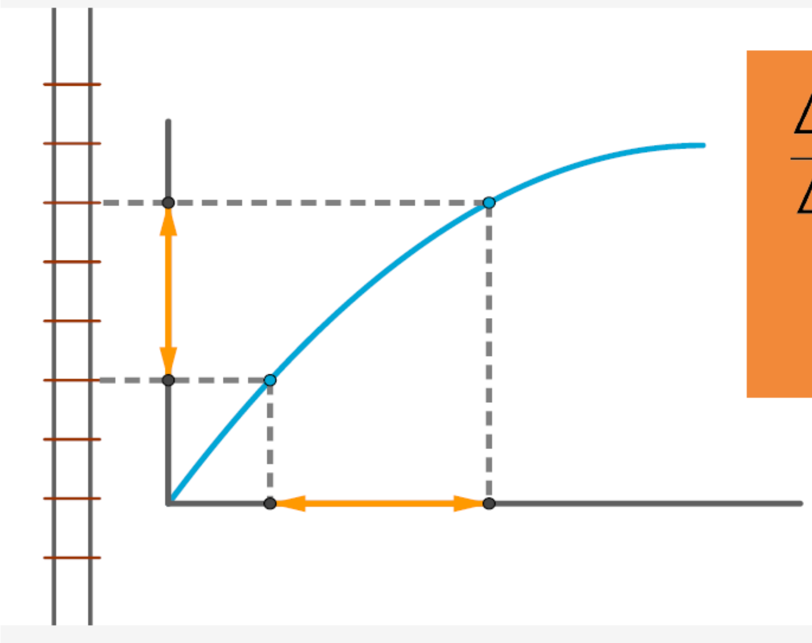
- The differentiation as a mathematical action is linked **with the optimization process**
- The differentiation is related **with the slope** of a function; in case of a straight track the slope of the function (the rate of its change with distance or with time) is constant; whereas in the case of a non straight track the slope changes from point to point. In the latter case we calculate the slope of a point!!!
- The differentiation is linked with the calculation of very important quantities e.g. the speed and the acceleration of a body

Definition of the Differentiation



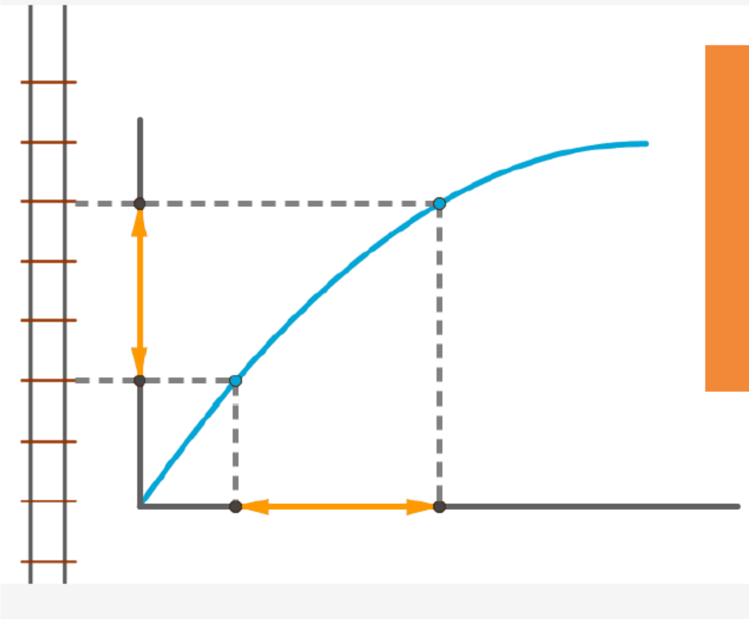


Definition of the Differentiation



$$\frac{\Delta x}{\Delta t} = \textbf{Average speed}$$

$$\approx \text{Speed at } t$$



Speed at t

$$= \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

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Definition of the Differentiation

Limits

- Slope = $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$
- Speed = $\lim\limits_{\Delta t \to 0} \dfrac{\Delta x}{\Delta t}$

Slope and speed are derivatives!

- Slope at position $x = a$: $\dfrac{dy}{dx}(a)$
- Speed at time $t = a$: $\dfrac{dx}{dt}(a)$

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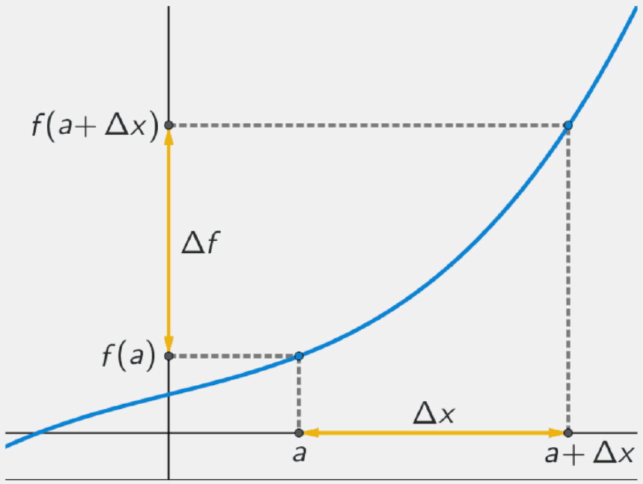
Definition of the Differentiation

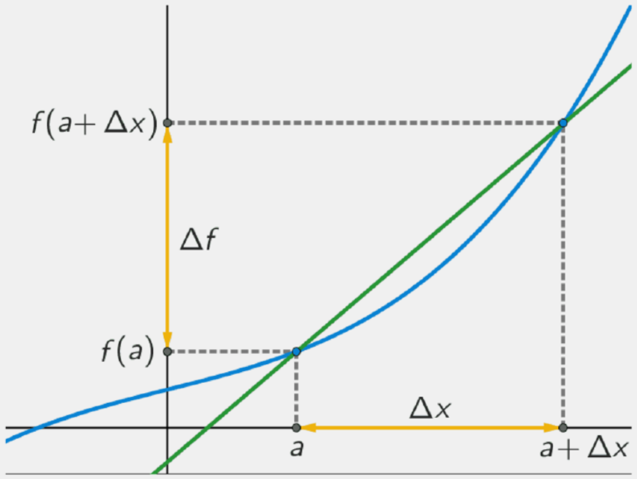
Differentiation – the definition



Difference quotient:

$$\frac{f(a+\Delta x)-f(a)}{\Delta x}$$

Differentiation – geometrical meaning

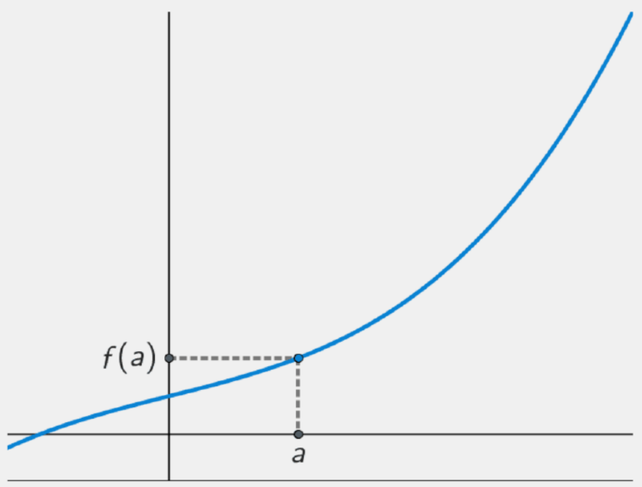


Slope of line *PQ*:

$$\frac{f(a+\Delta x)-f(a)}{\Delta x}$$

Definition of the Differentiation

Differentiation – the definition



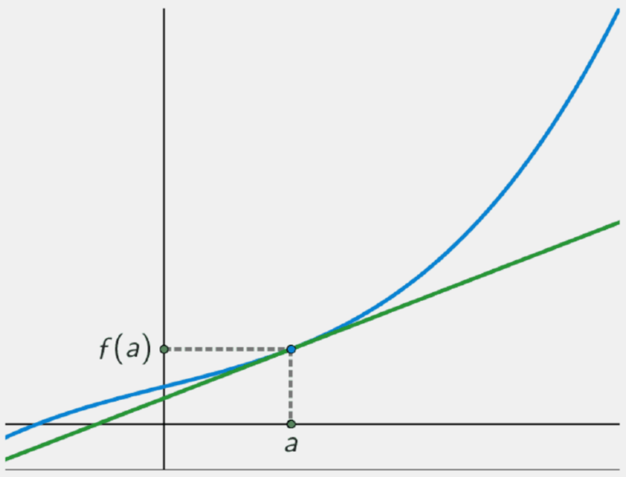
The derivative of f at $x = a$:

$$\lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

Notation:

$f'(a)$ or $\frac{df}{dx}(a)$

Differentiation – geometrical meaning



Slope of line PQ:

$$\frac{f(a + \Delta x) - f(a)}{\Delta x}$$

In limit:

Tangent line at *P*

Slope: $f'(a)$

Definition of the Differentiation

Limits

- Slope = $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$
- Speed = $\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$

Slope and speed are derivatives!

- Slope at position $x = a$: $\frac{dy}{dx}(a)$
- Speed at time $t = a$: $\frac{dx}{dt}(a)$

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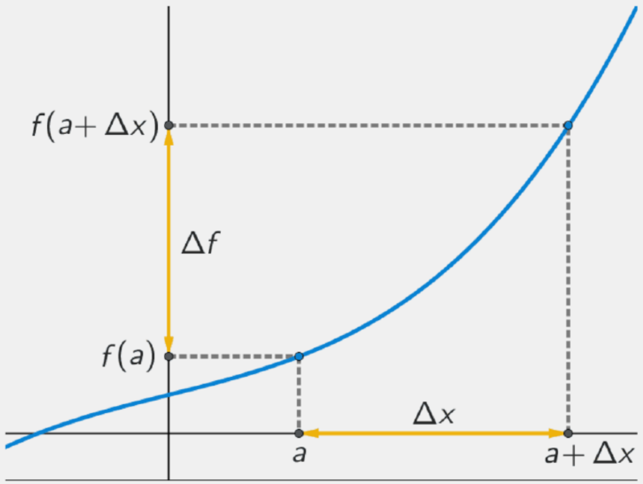
Definition of the Differentiation

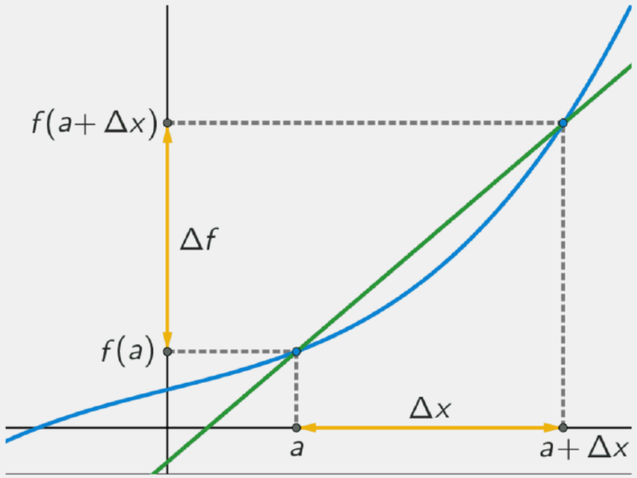
Differentiation – the definition



Difference quotient:

$$\frac{f(a+\Delta x)-f(a)}{\Delta x}$$

Differentiation – geometrical meaning

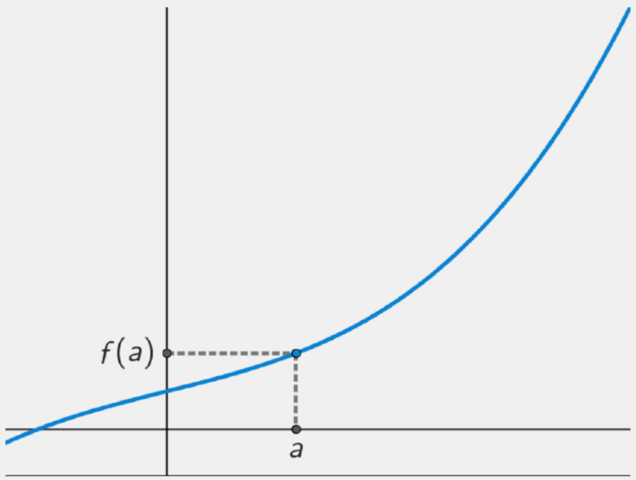


Slope of line *PQ*:

$$\frac{f(a+\Delta x)-f(a)}{\Delta x}$$

Definition of the Differentiation

Differentiation – the definition



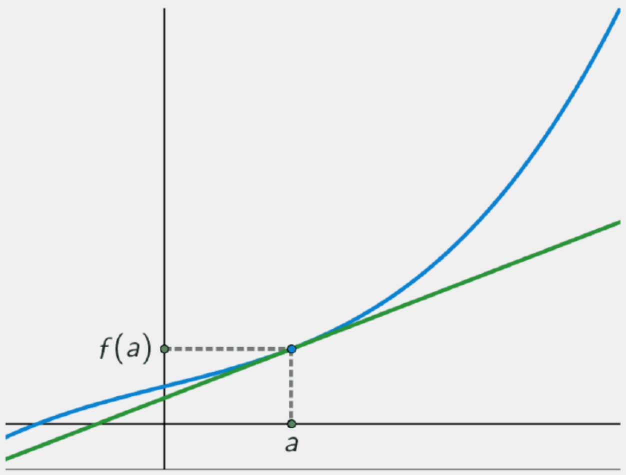
The derivative of f at $x = a$:

$$\lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

Notation:

$f'(a)$ or $\frac{df}{dx}(a)$

Differentiation – geometrical meaning



Slope of line PQ:

$$\frac{f(a + \Delta x) - f(a)}{\Delta x}$$

In limit:
Tangent line at *P*
Slope: $f'(a)$

Standard Derivatives & Rules of Calculation

- First, you learn **the derivatives of the standard functions**. Second, you learn **rules to calculate the derivative of combinations of standard functions**. Important rules of calculation are **the product rule and the chain rule**

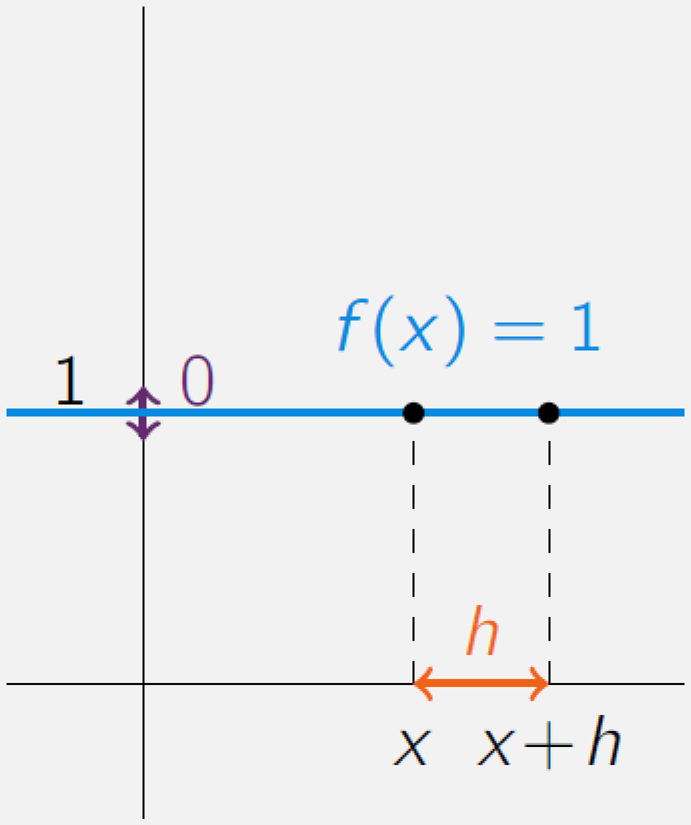
- The way we learn how to calculate the derivative of any function, independently how complicate could be, is the following:

  1. Learn the derivative of the standard functions
  2. Apply them and in combination of few rules of calculation, derive the derivatives of combinations of these functions

Standard Derivatives & Rules of Calculation

$$f(x) = x^0 = 1$$

$$\frac{f(x+h)-f(x)}{h} = \frac{1-1}{h} = \frac{0}{h} = 0$$

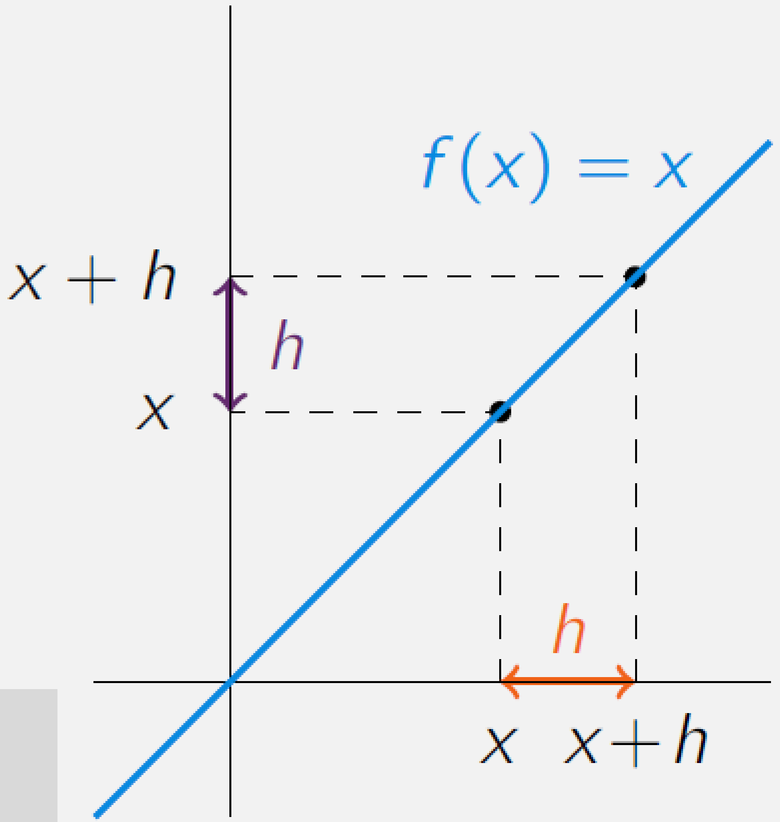


$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} 0 = 0$$

Standard Derivatives & Rules of Calculation

$$f(x) = x$$

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h) - x}{h} = \frac{h}{h} = 1$$



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 1 = 1$$

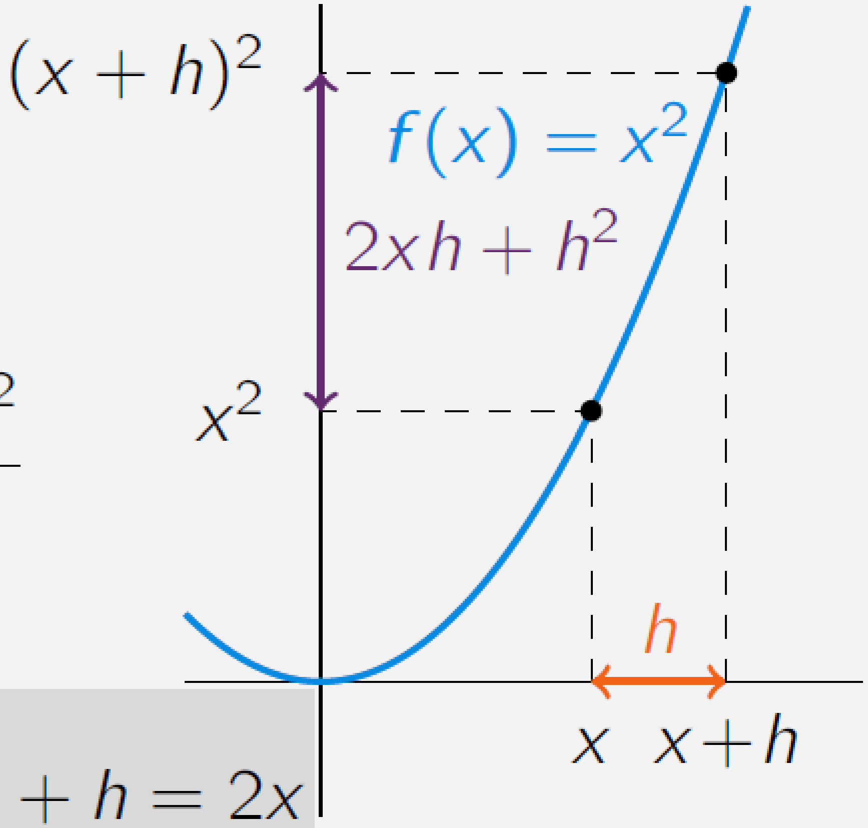
Standard Derivatives & Rules of Calculation

$$f(x) = x^2$$

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2 - x^2}{h}$$

$$= \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= 2x + h$$



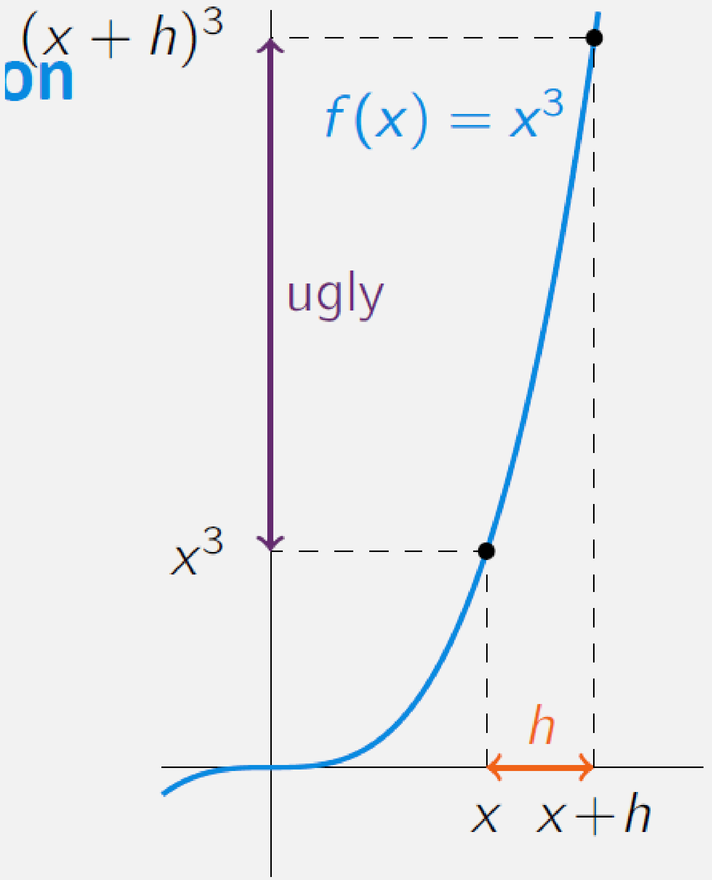
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} 2x + h = 2x$$

Standard Derivatives & Rules of Calculation

$$f(x) = x^3$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\text{ugly}}{h}$$

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x \cdot x^2) \\ &= \left(\frac{d}{dx}x\right) \cdot x^2 + x\left(\frac{d}{dx}x^2\right) \\ &= 1 \cdot x^2 + x \cdot 2x \\ &= 3x^2 \end{aligned}$$



Standard Derivatives & Rules of Calculation

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}\frac{1}{x} = \frac{d}{dx}x^{-1} = -x^{-2} = -\frac{1}{x^2}$$

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}x^\pi = \pi x^{\pi-1}$$

Attention: the highlighted formula is valid only for superscripts that are constants

Standard Derivatives & Rules of Calculation

$$\frac{d}{dx}(2 + 4x + 5x^2 + 7x^3 - x^4) =$$

$$= 2 \cdot 0 + 4 \cdot 1 + 5 \cdot 2x + 7 \cdot 3x^2 - 4x^3$$

$$= 4 + 10x + 21x^2 - 4x^3$$

Standard Derivatives & Rules of Calculation

Regarding the derivatives of functions of the type of a^x please remember the following:

$$x = \exp(\ln x) \Leftrightarrow a = \exp(\ln a)$$

$$\Leftrightarrow a^x = \exp[x \ln a] \Leftrightarrow (a^x)' = \exp[x \ln a]$$

$$\cdot \ln a \Leftrightarrow \boxed{(a^x)' = a^x \cdot \ln a} \quad \text{e.g. } (e^x)' = e^x$$

Standard Derivatives & Rules of Calculation

S. No.	f (x)	f′ (x)
1	x^n	nx^{n-1}
2	Sin x	Cos x
3	Cos x	-Sin x
4	Tan x	Sec 2x
5	Cot x	-Cosec2x
6	Sec x	Sec x Tan x
7	Cosec x	-Cosec x Cot x
8	log x	1/x
9	a constant	Zero
10	e^x	e^x
11	a^x	$a^x \log a$
12	√x	1 / (2√x)

Basic Derivatives Rules

Constant Rule: $\frac{d}{dx}(c) = 0$

Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Sum Rule: $\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$

Difference Rule: $\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x)+g(x)f'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$

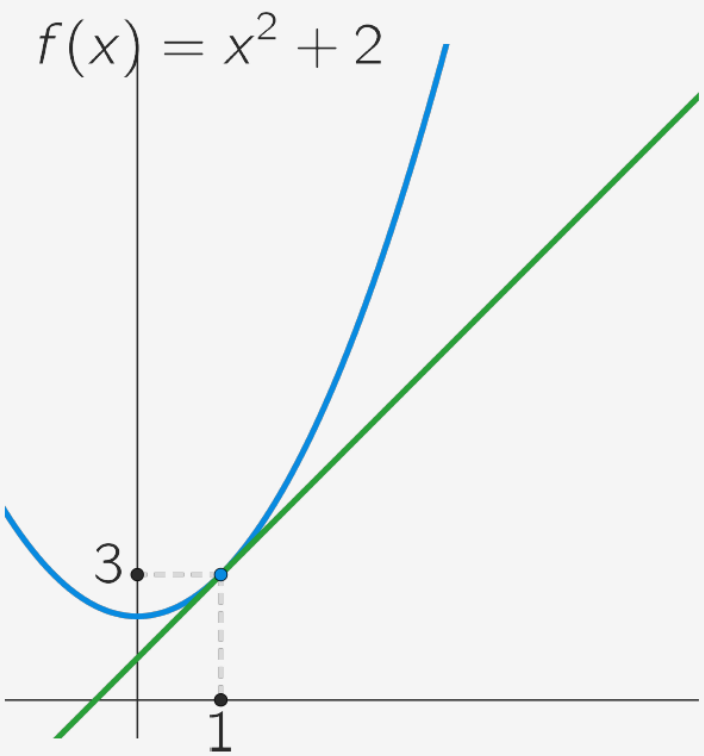
Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

Standard Derivatives & Rules of Calculation

- We know that **the derivative of a function at a point corresponds to the tangent line at this point of the function's graph**
- It is very important in order to realize the properties of a function, around a point, to derive the equation of this line, $y=a*x+b$

Standard Derivatives & Rules of Calculation

The tangent line – example

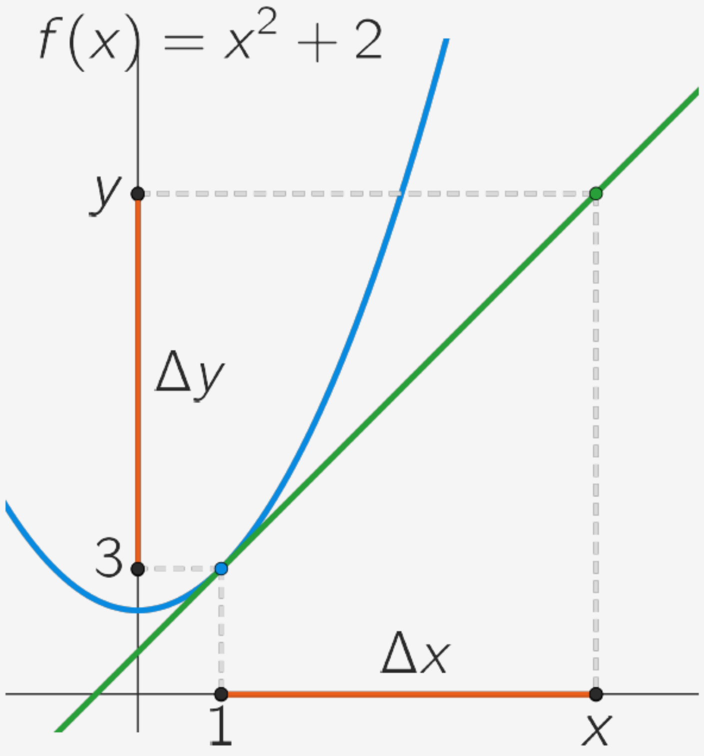


Slope $= f'(1) = 2$

$$f'(x) = \frac{d}{dx}(x^2 + 2) = 2x$$

Standard Derivatives & Rules of Calculation

The tangent line – example



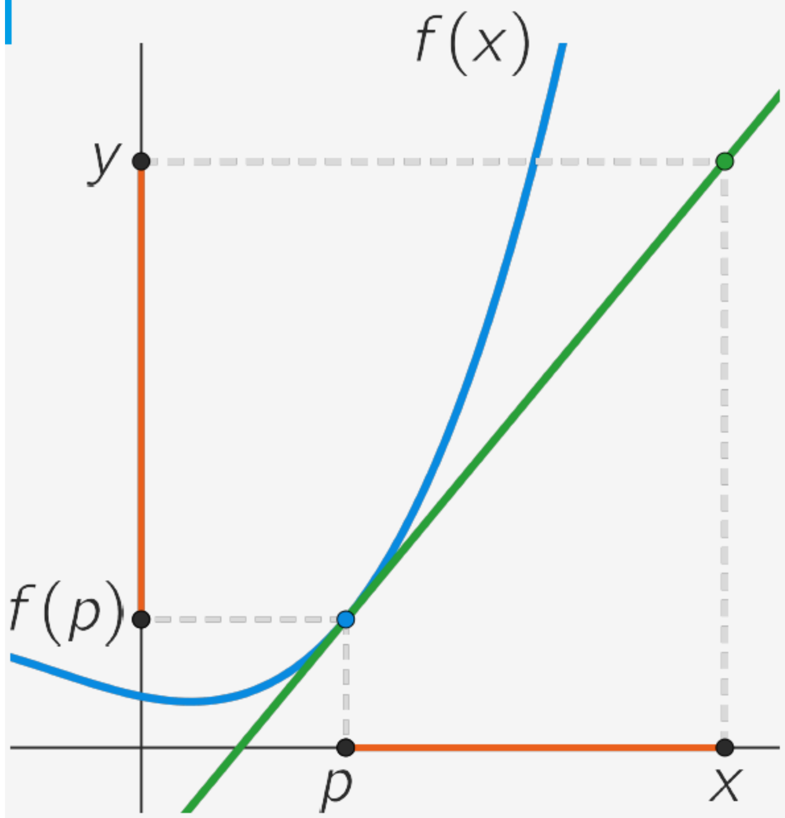
Slope $= f'(1) = 2$

$$2 = \frac{\Delta y}{\Delta x} = \frac{y-3}{x-1}$$

$$y - 3 = 2(x+1)$$

Standard Derivatives & Rules of Calculation

The tangent line



Tangent line:

- slope $f'(p)$
- through $(p, f(p))$

$$f'(p) = \frac{y - f(p)}{x - p}$$

Equation tangent line:

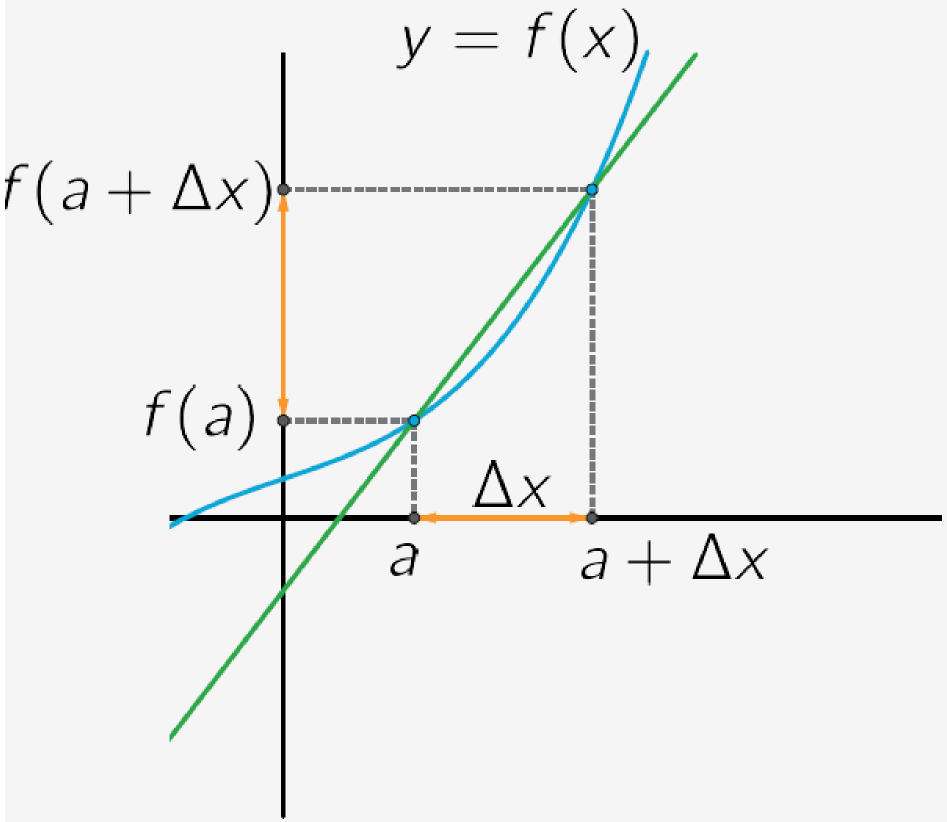
$$y = f'(p)(x - p) + f(p)$$

Standard Derivatives & Rules of Calculation

Non Differentiable Functions

- Can we differentiate any function? Can we differentiate any function at any point? **The answer to all of these questions is no!** What is the 'wrong' thing with the non differentiable functions?
- A function **is not differentiable at a point, when a tangent line passes through this point with a finite slope does not exist**

Standard Derivatives & Rules of Calculation



The function f is differentiable at a if

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

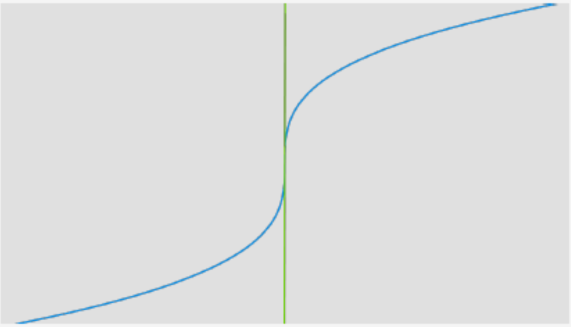
exists.

Standard Derivatives & Rules of Calculation

Nondifferentiable functions

Functions f that are not differentiable $x = a$:

- f is discontinuous at $x = a$.
- The graph of f has a kink at $x = a$.
- The graph of f has a vertical tangent line at $x = a$.



- More ‘exotic’ functions, e.g. $x \sin(\frac{1}{x})$.

Standard Derivatives & Rules of Calculation

- An example of non-differentiable function is: $f(x) = x^{1/3}$ at zero. Can you explain why?
- Another example of a non-differentiable function is: $f(x) = abs(x)$. Can you explain why?

Standard Derivatives & Rules of Calculation

- **For optimization** it is important **to be able to find the minima and maxima** of a function
- There are two types of maxima and minima within a graph: **the global ones** (the highest or the lowest value across the function's domain) or **the local ones** (the lowest or the highest one around a specific point)
- **Graph contains points that** can be called either (a) critical points (where the derivative is zero); (b) singular points (where the derivative is not defined); or (c) boundary points (where the derivative is not zero) – <u>Among these points we should check for maxima or minima points!!</u>

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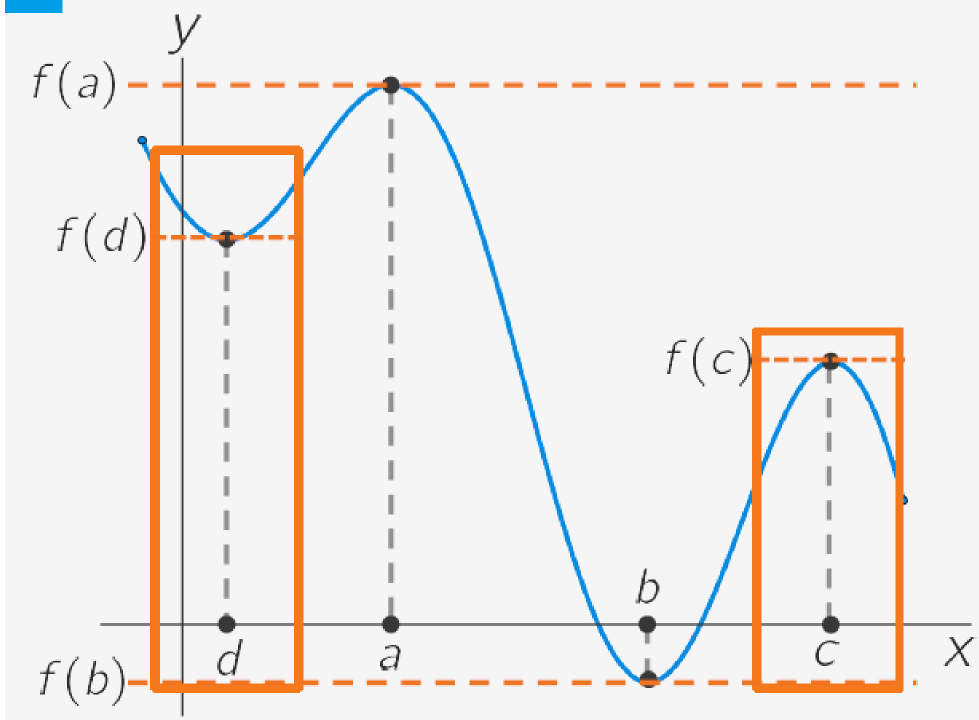
Standard Derivatives & Rules of Calculation

Minima and maxima



Global

maximum at $x = a$

minimum at $x = b$

Local

maximum at $x = c$

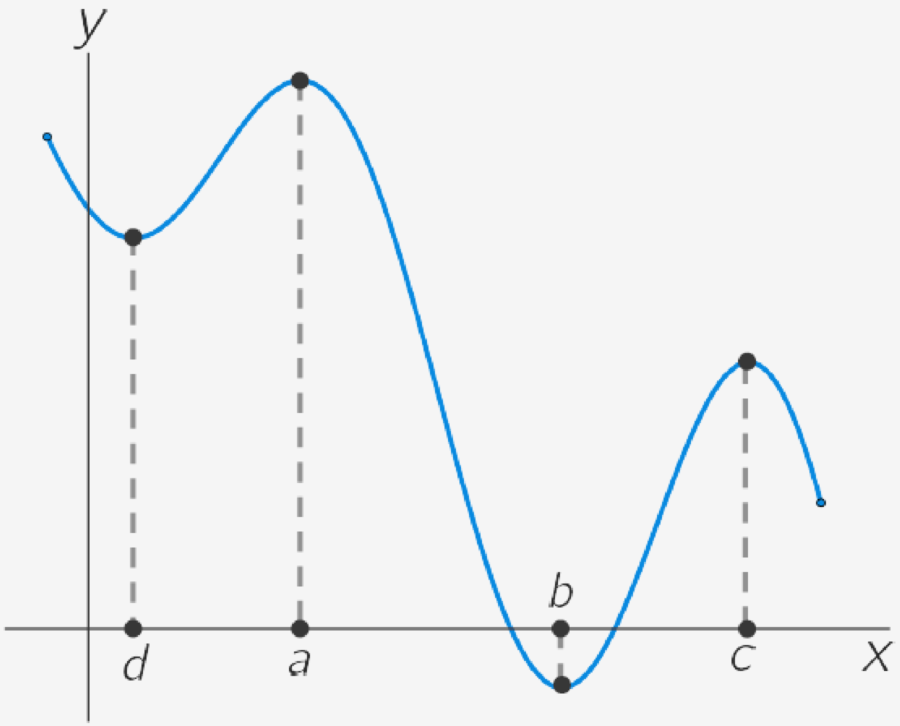
and $x = a$

minimum at $x = d$

and $x = b$

Standard Derivatives & Rules of Calculation

Minima and maxima



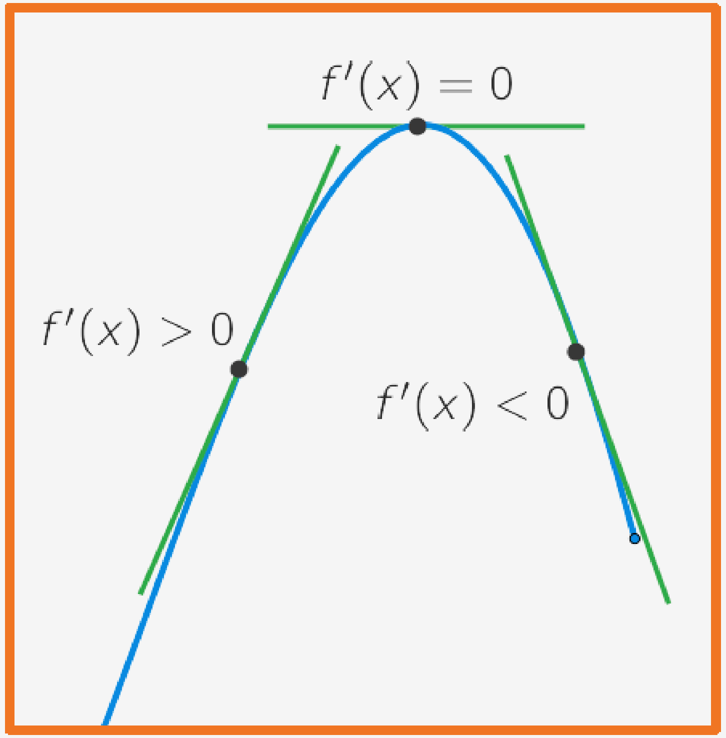
$f'(x) > 0 \Rightarrow$ increase

$f'(x) < 0 \Rightarrow$ decrease

at extrema: $f'(x) = 0$

Standard Derivatives & Rules of Calculation

Minima and maxima



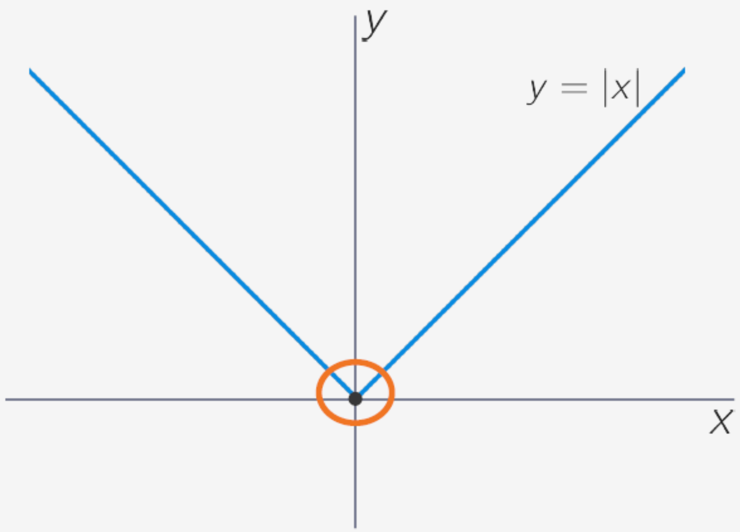
$f'(x) > 0 \Rightarrow$ increase

$f'(x) < 0 \Rightarrow$ decrease

at extrema: $f'(x) = 0$

Standard Derivatives & Rules of Calculation

Non-differentiable functions



$f(x) = |x|$

minimum at $x = 0$

$f'(0)$ **does not exist!**

Boundary points

Minima at feet

f' exists

$f' \neq 0$ at feet

Boundary points

Standard Derivatives & Rules of Calculation

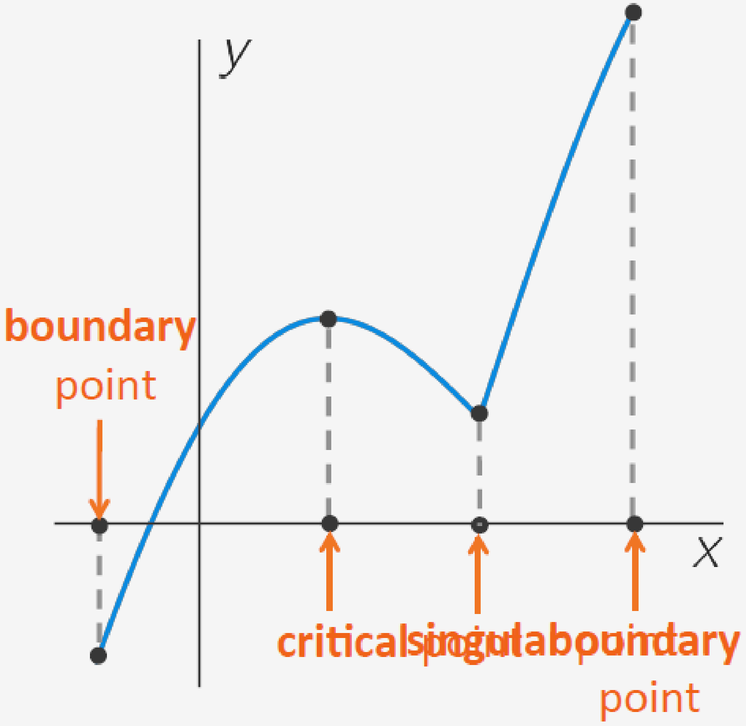
Finding minima and maxima

Given

- function f;
- point a such that $f(a)$ is local extremum;

Then

- $f'(a) = 0$,
- or $f'(a)$ does not exist,
- or a is a boundary point.



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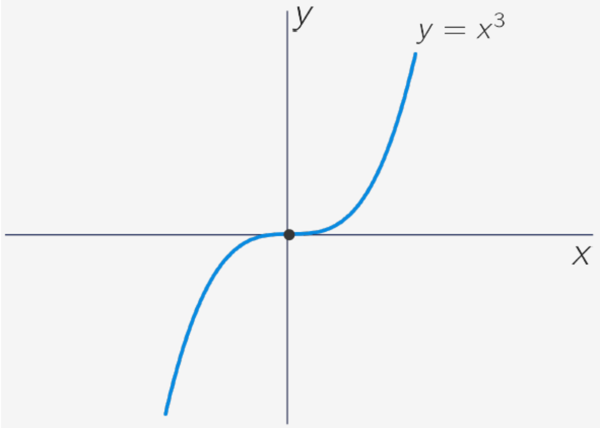
Standard Derivatives & Rules of Calculation

Warning!



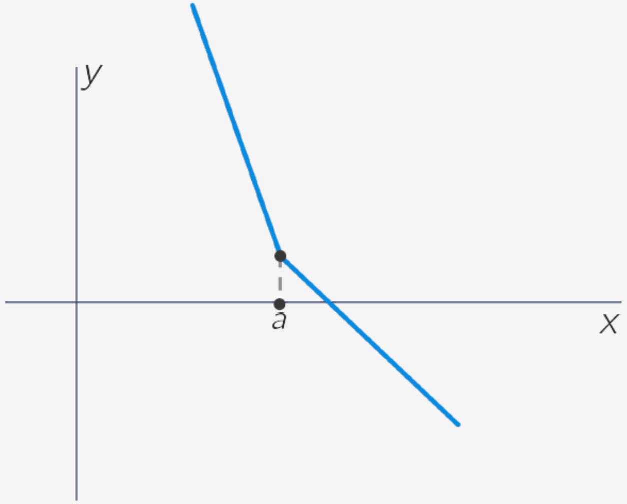
$f(x) = x^3$

$f'(x) = 3x^2$

$f'(0) = 0$

No extremum at $x = 0$

Warning!



Singular point $x = a$

No extremum at $x = a$

Standard Derivatives & Rules of Calculation

Finding local extrema

1. Find:
   - critical points: $f'(x) = 0$
   - singular points : $f'(x)$ does not exist
   - boundary points

2. Check at each at these points:
   - local minimum?
   - local maximum?
   - *neither*?

Higher order differentiation

If f is a differentiable function and $f'(x) = \frac{df}{dx}$ its first derivative in respect to the variable x,

then

the derivative of $f'(x)$ (if it exists) is denoted as

$$f''(x) = \frac{d^2 f(x)}{dx^2}$$

and is called <u>second derivative</u> of f.

Higher order differentiation

The same way, the derivative of the second derivative (if it exists) is denoted as

$$f^{(3)}(x) = \frac{d^3 f(x)}{dx^3}$$

And is called the <u>third derivative</u> of *f*.

Continuing this process, from the *(v-1)*-th derivative of *f* we can derive the *v*-th derivative of *f*.

The *v*-th derivative is called **derivative of order *v***and is denoted as $f^{(v)}(x) = \frac{d^v f(x)}{dx^v}$.

Higher order differentiation

Example:

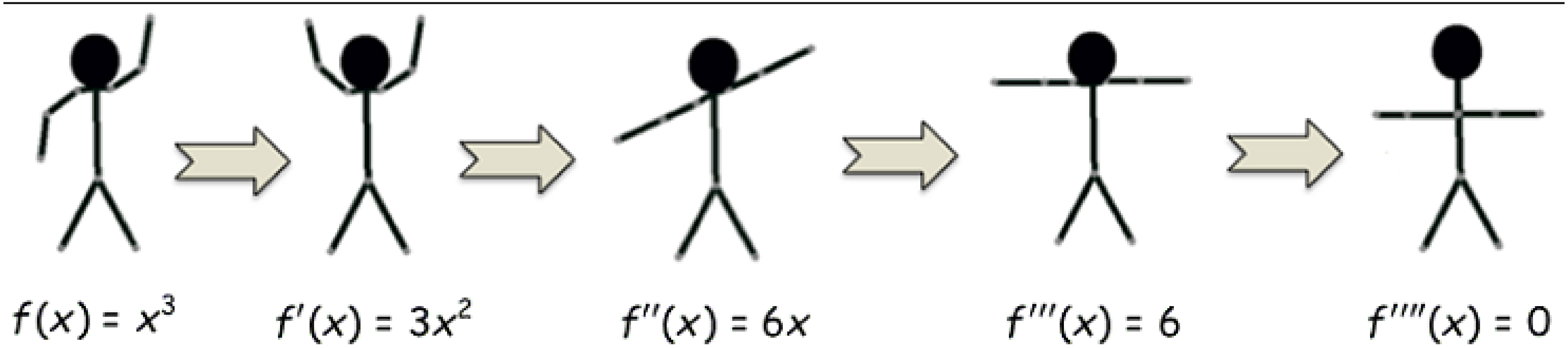
Assume *f(x)*=$x^3 - 3x^2 + 2$.

Then

- $f'(x) = 3x^2 - 6x$,
- $f''(x) = 6x - 6$,
- $f^{(3)}(x) = 6$ and
- $f^{(4)}(x) = 0$.

Higher order differentiation

The Derivatives Dance



Differentiation

- If you want to learn more **about differentiation** please check the following video lectures:

1. https://www.youtube.com/watch?v=xd703YLsLAY&feature=youtu.be
2. https://www.youtube.com/watch?v=Rpum6FRM2UU&feature=youtu.be