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Department of Electronic Engineering

Faculty of Applied Sciences

Centre for Plasma Physics & Lasers (CPPL)

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PowerLaPs
High Power Laser Plasma Physics

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Director CPPL

PowerLaPs

Rethymno 02-13 July 2018

ΙΔΡΥΜΑ ΚΡΑΤΙΚΩΝ ΥΠΟΤΡΟΦΙΩΝ IKY

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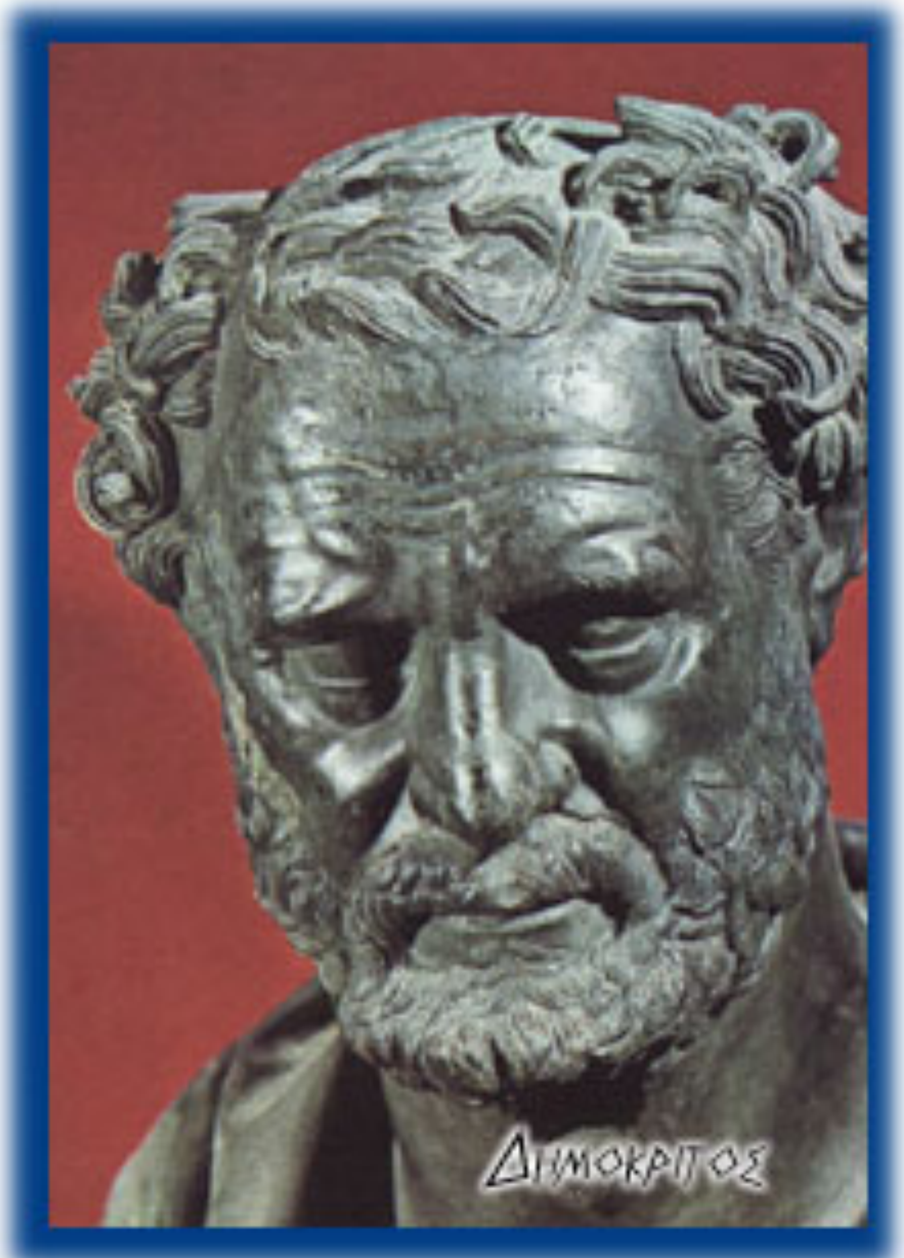

"In principle our knowledge is limited as it is in the far interior of matter" , 4th century bc

PLASMA & LASER TECHNOLOGY

ENJOY THE JOURNAY

Introduction to plasma physics

Definition of Plasma, the “fourth state of matter”

Particle motions in a plasma

Plasma as a fluid

Waves in plasmas

Plasma equilibrium

Kinetic theory

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PowerLaPs
High Power Laser Plasma Physics

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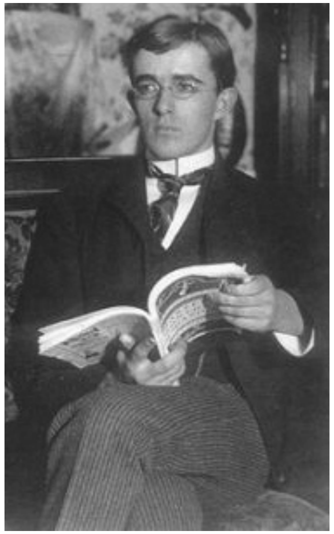
Plasma is a partially or completely ionised medium, presents a collective behavior (unlike the ideal gas). It is considered as the fourth state of matter (unlike the ideal gas)

It is named "plasma" «πλάσμα» by I. Langmuir and L. Tonks

Plasma should not contain more than 1% neutral atoms or molecules



- It is a magnificent material

- It is very selfish, does not like external influences but likes group work

- It has mind on its own

• Plasma is the "fourth state" of matter

Consists of electrons, ions and neutral atoms

Index of refraction < 1

99% of matter of the visible Universe is in plasma state

is any ionised gas plasma?

• Definition of plasma

A quasineutral gas of charged and neutral particles which exhibits collective behavior

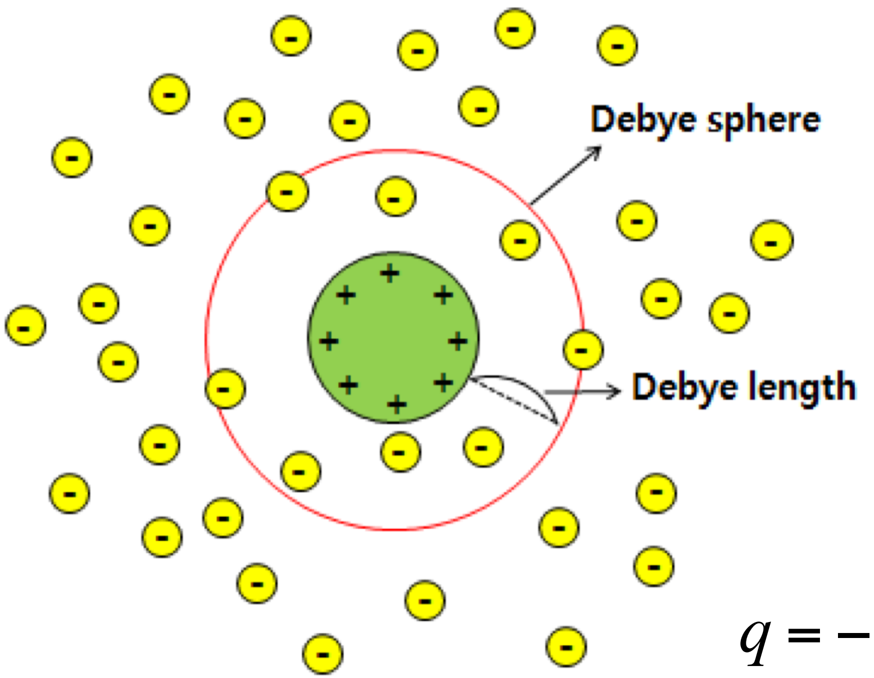
Quasineutral:

The plasma is neutral enough so that one can take $n_i \sim n_e \sim n$

Collective:

Plasma charges move around and generate local concentrations of positive and negative charge and therefore electric fields. Also, motion of charges generates currents, thus magnetic fields. These fields affect the motion of other particles far away.

Debye shielding



Poisson's eq.

$$\nabla^2\phi = \frac{d^2\phi}{dx^2} = -\frac{e}{\varepsilon_o}\left(n_i - n_e\right)$$

$$n_i = n_\infty \qquad Z=1$$

Electron distribution function

$$f(u) = C\exp\left[-\left(\frac{1}{2}mu^2 + q\phi\right)/kT_e\right]$$

$$q = -e$$

$$n_e(\phi \to 0) = n_\infty \qquad n_e = \int_0^\infty f(u)\,du = n_\infty \exp\left(\frac{e\phi}{kT_e}\right)$$

$$\frac{d^2\phi}{dx^2} = \frac{e}{\varepsilon_o}n_\infty\left\{\left[\exp\frac{e\phi}{kT_e}\right] - 1\right\} = \frac{e}{\varepsilon_o}n_\infty\left[\frac{e\phi}{kT_e} + \frac{1}{2}\left(\frac{e\phi}{kT_e}\right)^2 + \ldots\right]$$

Taylor expansion

Debye shielding

$$\frac{d^2\phi}{dx^2} = \frac{e^2 n_\infty}{\varepsilon_o k T_e}\phi, \lambda_D \equiv \left(\frac{\varepsilon_o k T_e}{n e^2}\right)^{1/2}$$

$$\phi = \phi_o \exp\left(-\frac{x}{\lambda_D}\right)$$

$$\lambda_D = 69\left(\frac{T_e}{n}\right)^{1/2} m, T(^oK)$$

$$\lambda_D = 7430\left(\frac{kT_e}{n}\right)^{1/2} m, kT(eV)$$

λ_D Debye length

The plasma parameter

Number of particles in the Debye sphere

$$N_D = n\frac{4}{3}\pi\lambda_D^3 = 1.38x10^6\frac{T^{3/2}}{n^{1/2}}, T(^oK)$$

Collective behavior requires:

$$\lambda_D << L$$

$$N_D >> 1$$

Plasma	n_e (m^{-3})	T (K)	B (T)	λ_D (m)	N_D	ω_p (s^{-1})	ν_{ee} (s^{-1})	ω_c (s^{-1})	r_L (m)
Gas discharge	10^{16}	10^{4}	—	10^{-4}	10^{4}	10^{10}	10^{5}	—	—
Tokamak	10^{20}	10^{8}	10	10^{-4}	10^{8}	10^{12}	10^{4}	10^{12}	10^{-5}
Ionosphere	10^{12}	10^{3}	10^{-5}	10^{-3}	10^{5}	10^{8}	10^{3}	10^{6}	10^{-1}
Magnetosphere	10^{7}	10^{7}	10^{-8}	10^{2}	10^{10}	10^{5}	10^{-8}	10^{3}	10^{4}
Solar core	10^{32}	10^{7}	—	10^{-11}	1	10^{18}	10^{16}	—	—
Solar wind	10^{6}	10^{5}	10^{-9}	10	10^{11}	10^{5}	10^{-6}	10^{2}	10^{4}
Interstellar medium	10^{5}	10^{4}	10^{-10}	10	10^{10}	10^{4}	10^{-5}	10	10^{4}
Intergalactic medium	1	10^{6}	—	10^{5}	10^{15}	10^{2}	10^{-13}	—	—

Source: Chapter 19: *The Particle Kinetics of Plasma*
http://www.pma.caltech.edu/Courses/ph136/yr2004/

Plasma Criteria

$$\lambda_D << L$$

$$N_D >> 1$$

$$\omega\tau > 1$$

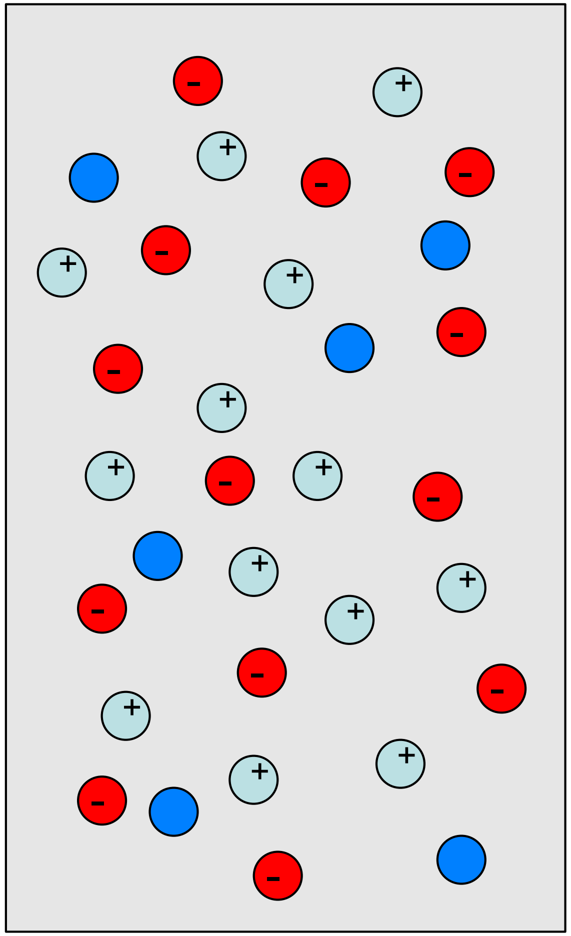
ω is the frequency of typical plasma oscillations

τ is the mean time between collisions with neutral atoms

$n_o = 10^{20}\ m^{-3}$, $T = 1$ keV, $\lambda_D = 20\ \mu m$

$n_o = 10^{8} m^{-3}$, $T = 10$ eV, $\lambda_D = 3\ m$

For a plasma- Saha equation



i.e. hydrogen plasma in thermodynamic equilibrium

$$N_e = N_p \quad N_n$$

$$n_e = n_p \quad n_n$$

Total density:

$$n = n_n + n_e = n_n + n_p$$

gas temperature: $T^o K$

ionisation energy: E_i

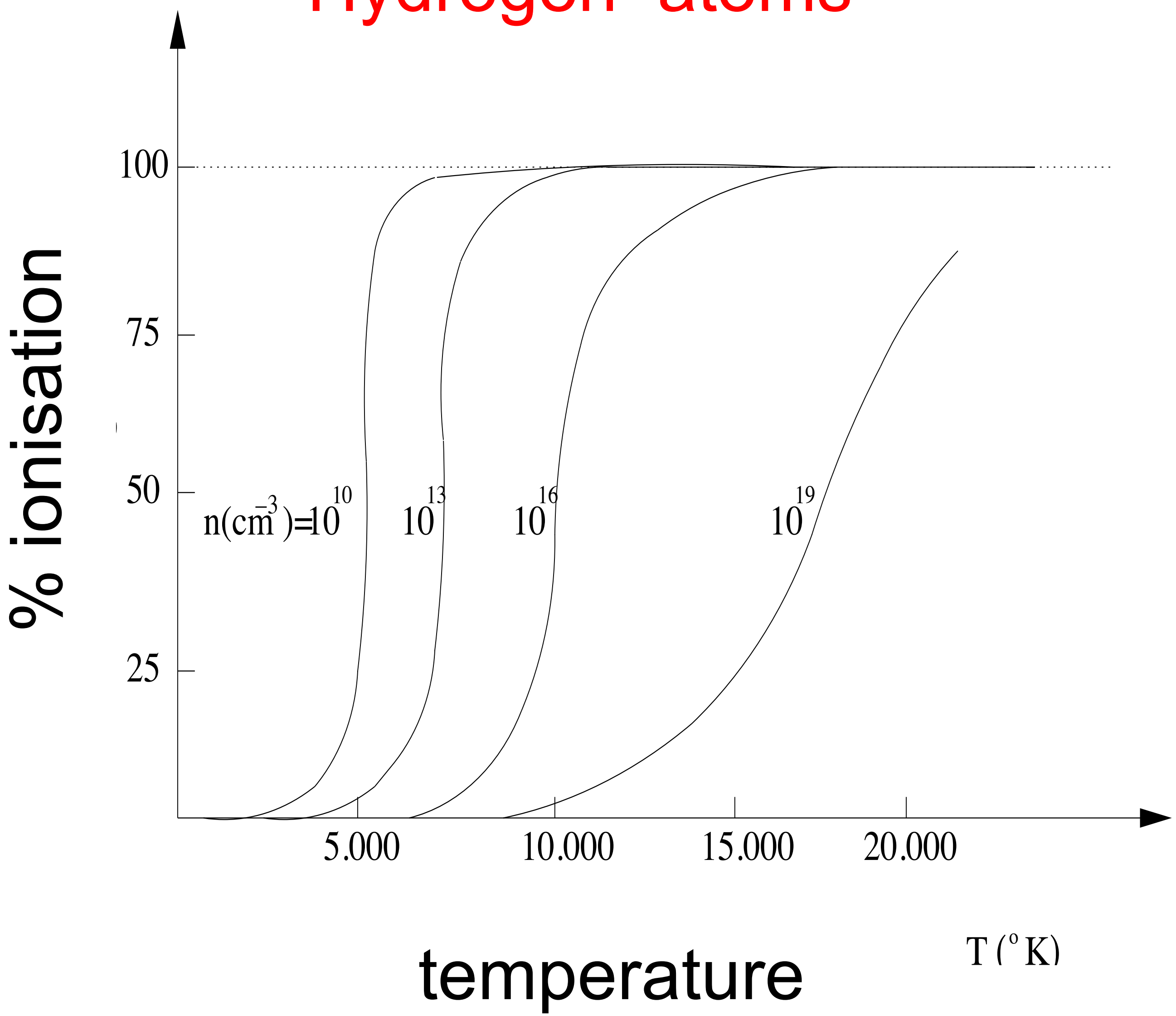
Megh Nad Saha
“Saha equation”

$$\frac{n_i}{n_n} = \frac{(2\pi m_e kT)^{3/2}}{nh^3} e^{-\frac{E_i}{kT}} = 2.4x10^{15} \frac{T^{3/2}}{n_i} e^{-\frac{E_i}{kT}}$$

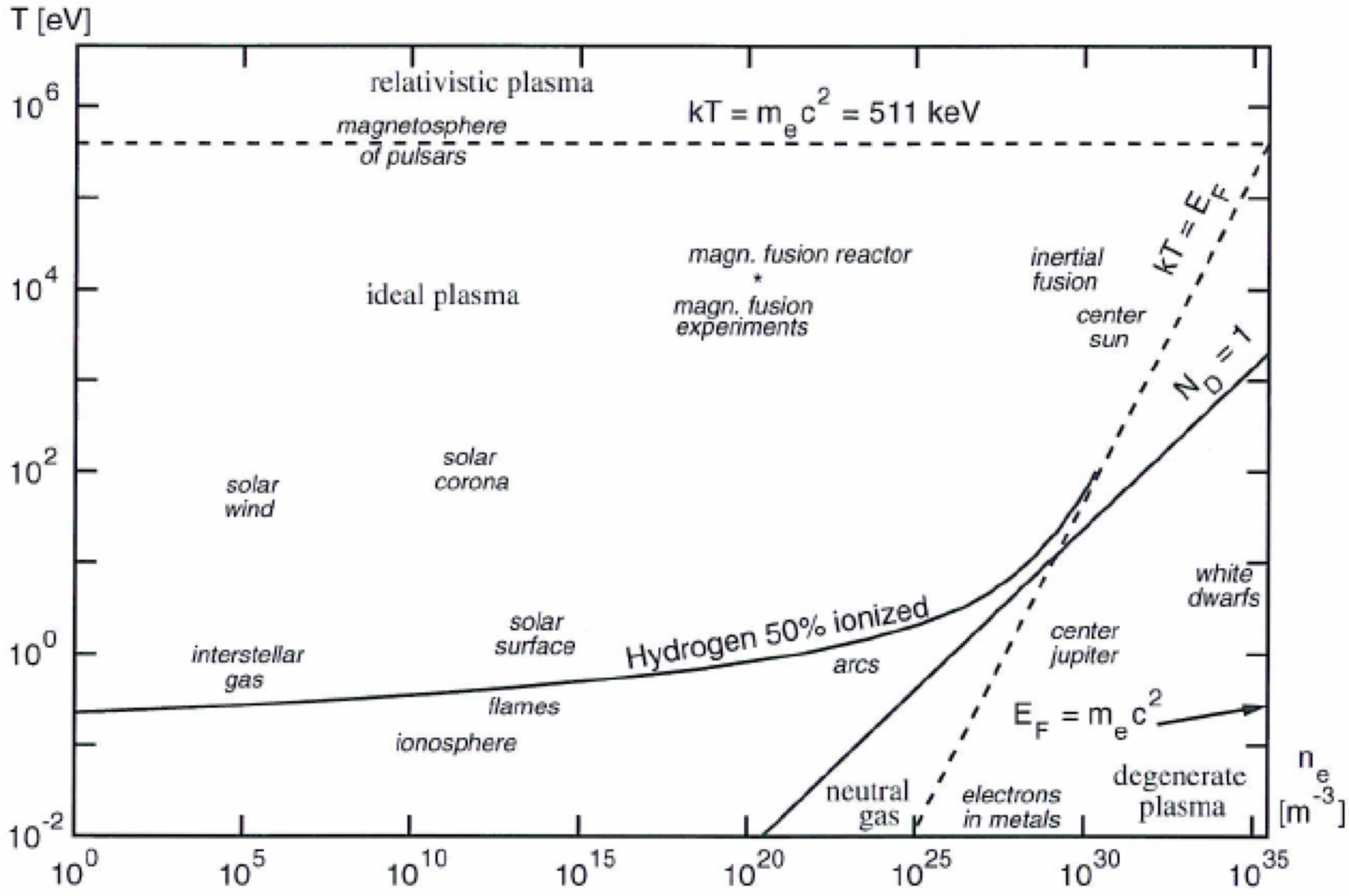
T=300^{o}K, E_i=14,5eV (nitrogen), n_n=3x10^{25}m^{-3}

$$\frac{n_i}{n_n} \approx 10^{-122}$$

Ionisation rate
Hydrogen atoms

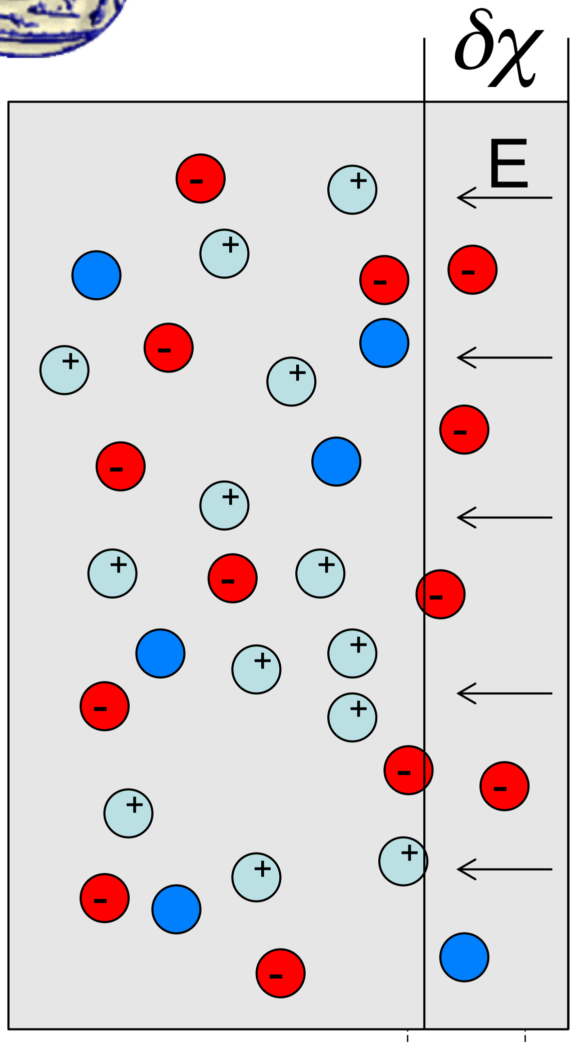


Kind of plasma	Density n [cm-3]	Temperature T [K]
Electrical discharges	$\sim 10^{10}$	~10,000
Earth's ionosphere	$\sim 10^{6}$	~1,000
Solar wind	~10	Te ~ 500,000, Ti ~ 100,000
Solar corona	$\sim 10^{9}$	~10,000,000
Tokamak	$\sim 10^{15}$	~100,000,000
White dwarfs	$\sim 10^{30}$	~10,000
Hydrogen bomb	$\sim 10^{30}$	~100,000,000,000
Interplanetary space	$\sim 10^{-2} – 1$	~100
Laser plasma		



T [eV]
relativistic plasma
kT = m_e c^2 = 511 keV
magnetosphere of pulsars
magn. fusion reactor
magn. fusion experiments
inertial fusion
center sun
kT = E_F
N_D = 1
ideal plasma
solar wind
solar corona
white dwarfs
Hydrogen 50% ionized
interstellar gas
solar surface
arcs
center jupiter
flames
ionosphere
E_F = m_e c^2
neutral gas
electrons in metals
degenerate plasma
n_e [m^-3]
10^6
10^4
10^2
10^0
10^-2
10^0
10^5
10^10
10^15
10^20
10^25
10^30
10^35

Plasma frequency



Η εξίσωση κίνησης των ηλεκτρονίων είναι:

$$\sigma = n_o e \delta\chi$$

$$m_e \frac{d^2\delta\chi}{dt^2} = -eE = -\frac{n_o e^2 \delta\chi}{\varepsilon_o} \qquad E = \frac{\sigma}{\varepsilon_o} = \frac{V}{d}$$

Τα ηλεκτρόνια εκτελούν <u>αρμονική ταλάντωση</u> γύρω από την θέση ισορροπίας με συχνότητα :

$$\omega_p = \left(\frac{n_0 e^2}{m\varepsilon_0}\right)^{1/2}$$

Αν θεωρήσουμε και την κίνηση των ιόντων τότε

η συχνότητα πλάσματος είναι :

$$\omega_p = \left(\frac{n_0 e^2}{m_e\varepsilon_0} + \frac{n_0 e^2}{m_i\varepsilon_0}\right)^{1/2}$$

Some important plasma parameters

Plasma frequency $\omega_{pe} = \sqrt{\dfrac{n_e e^2}{\varepsilon_o m_e}} = 5.64 \times 10^4 n_e^{1/2} rad \sec^{-1}$

Critical density $n_c = \dfrac{\omega^2 m_e \varepsilon_o}{e^2} = 1.113 \times 10^{21} \left(\dfrac{1\mu m}{\lambda}\right)^2 cm^{-3}$

Debye length $\lambda_{De} = \left(\dfrac{\varepsilon_o k_B T_e}{n_e e^2}\right)^{1/2} = 2.35 \times 10^{-8} \left(\dfrac{T_e}{1eV}\right)^{1/2} \left(\dfrac{10^{21}}{n_e (cm^{-3})}\right)^{1/2} cm$

Debye number $N_D = n \dfrac{4}{3} \pi \lambda_D^3 = 1.38 \times 10^6 T^{3/2} / n^{1/2}$

Average number of electrons in a plasma contained within a Debye sphere

The Plasma as a Fluid

In a real plasma the positions and motions of the particles (electrons, ions and neutral atoms) are determined by randomness due to thermal effects and unexpected internal or external perturbations. The **E** and **B** fields are therefore also determined by the motions of the particles in the plasma and the currents that are generated due to the external or internal perturbations.

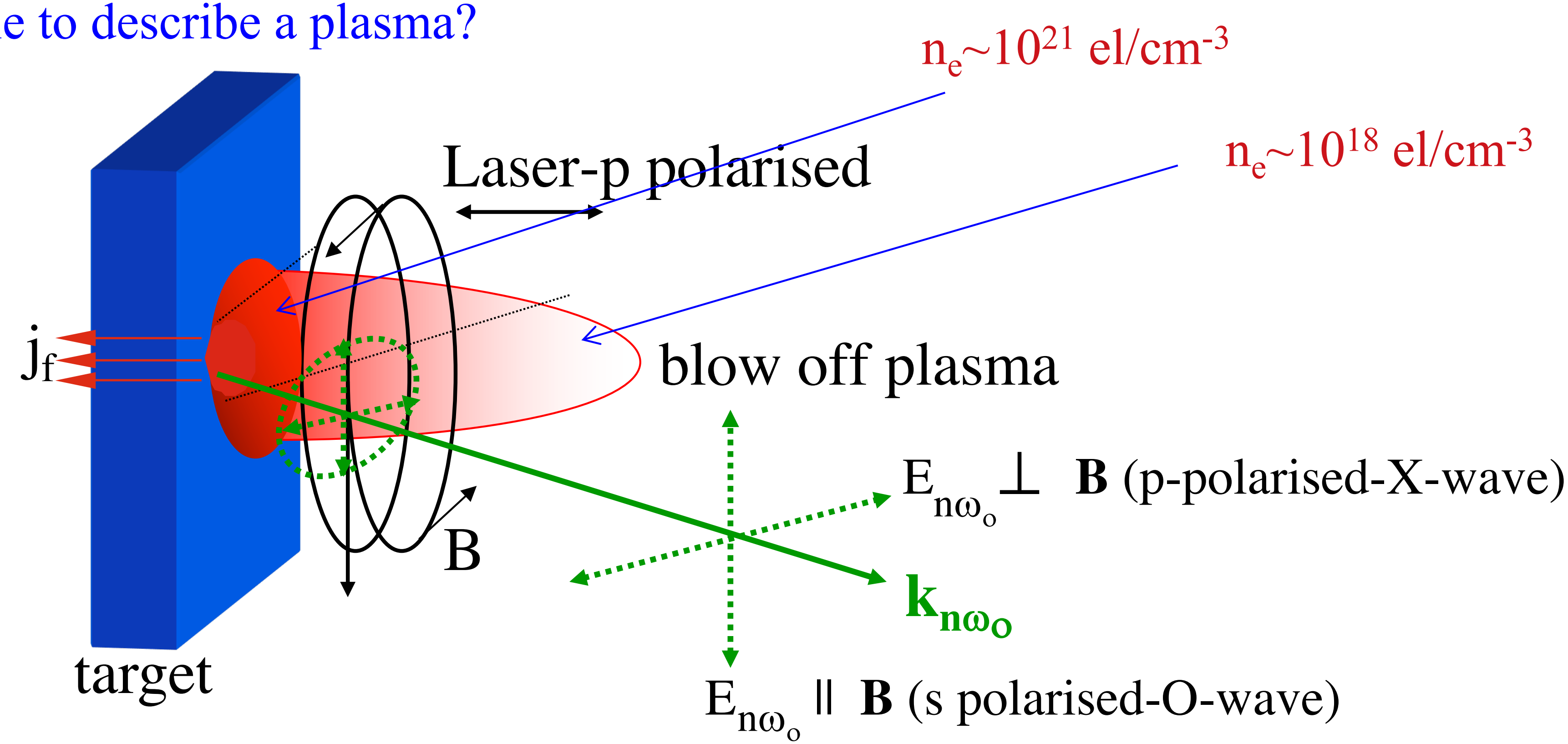
Since the density of the plasma can vary from $\sim 10^{12}$ particles/cm^3 up to $\sim 10^{24}$ particles/cm^3, it is obvious that in order to study the plasma dynamics it is really **almost impossible** to follow each particle's trajectory in time and its interaction with the other particles and the time varying generated **E** and **B** fields.

Fortunately, most of the plasma phenomena for experimentally generated plasmas (i.e. using lasers or pulsed power devices such as Z or X-pinches or Tokamak, can be well described using fluid mechanics physics such as the plasma is fluid*. In this case the behaviour and identity of the individual particles is not taken into account and instead, the motion of fluid elements using the fluid equations adapted to the plasma conditions (and the fact that the fluid contains charged particles as well as **E** and **B** fields) are implemented.

*i.e., see the plasma generated by lasers and collisions in the next slides

The Plasma as a Fluid

Can we really use the eq. of motion of a single particle to describe a plasma?



A laser generated plasma is comprised of regions with high charge density as well as regions with lower charge density. Density scale-length is of the order of μm!

Within a few μm the charge density is dropped a few orders of magnitude!!

The Plasma as a Fluid - Maxwell's equations

In vacuum		In a medium
(1) $\nabla \cdot \boldsymbol{E} = \dfrac{\rho_f}{\varepsilon_o}$	Gauss's Law for electricity	$\nabla \cdot \boldsymbol{D} = \rho_f$
(2) $\nabla \cdot \boldsymbol{B} = 0$	Gauss's Law for magnetism	$\nabla \cdot \boldsymbol{B} = 0$
(3) $\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$	Faraday's Law of induction (επαγωγή)	$\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$
(4) $\nabla \times \boldsymbol{B} = \mu_o(\boldsymbol{j}_f + \varepsilon_o \dfrac{\partial \boldsymbol{E}}{\partial t})$	Ampere's Law	$\nabla \times \boldsymbol{H} = \boldsymbol{j}_f + \dfrac{\partial \boldsymbol{D}}{\partial t}$

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} = \varepsilon_o \boldsymbol{E} + \boldsymbol{P}$$

$$\boldsymbol{P} = \varepsilon_o \chi_\varepsilon \boldsymbol{E} \qquad \varepsilon = \left(1 + \chi_\varepsilon\right)\varepsilon_o$$

$$\boldsymbol{B} = \mu_m \boldsymbol{H}$$

Magnetic susceptibility ↓

$$\mu_m = \left(1 + \chi_m\right)\mu_o \qquad \boldsymbol{M} = \chi_m \boldsymbol{H}$$

The Plasma as a Fluid - Maxwell's equations

In equations (1) and (4) ρ_f and $\boldsymbol{J}_f$ are the "free" charge and current density. The "bound charge" and current densities arising from the polarisation and magnetisation of the plasma (like in a dielectric medium) are included in the quantities **D** and **H,** which are the **electric displacement field** and magnetic field **H** (same name as the magnetic field **B**) respectively.

The total charge density is $\rho_q = \rho_f + \rho_b$. So in a plasma the Gauss law would write:

$$\nabla \cdot \boldsymbol{D} = \rho_f \Leftrightarrow \nabla \cdot (\varepsilon_o \boldsymbol{E} + P) = \varepsilon_o (1 + \chi_\varepsilon) \boldsymbol{E}$$

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} = \varepsilon_o \boldsymbol{E} + \boldsymbol{P}$$

$$\boldsymbol{P} = \varepsilon_o \chi_\varepsilon \boldsymbol{E} \qquad \varepsilon = \left(1 + \chi_\varepsilon\right) \varepsilon_o$$

$$\mathbf{H} \equiv \frac{\mathbf{B}}{\mu_0} - \mathbf{M}. \qquad \boldsymbol{B} = \mu_o (\boldsymbol{H} + \boldsymbol{M})$$

The Plasma as a Fluid

Convective derivative

Definition of Convective Derivative:

A derivative taken with respect to a moving coordinate system, also called the Langrangian derivative, substantive derivative, or Stokes derivative. It is given by:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{\upsilon} \cdot \boldsymbol{\nabla}$$

where is the gradient operator and **υ** is the velocity of the fluid. This type of derivative is especially useful in the study of fluid mechanics. When applied to **υ**,

$$\frac{D\boldsymbol{\upsilon}}{Dt} = \frac{\partial \boldsymbol{\upsilon}}{\partial t} + (\boldsymbol{\nabla} \times \boldsymbol{\upsilon}) \times \boldsymbol{\upsilon} + \boldsymbol{\nabla}(\frac{1}{2}\boldsymbol{\upsilon}^2)$$

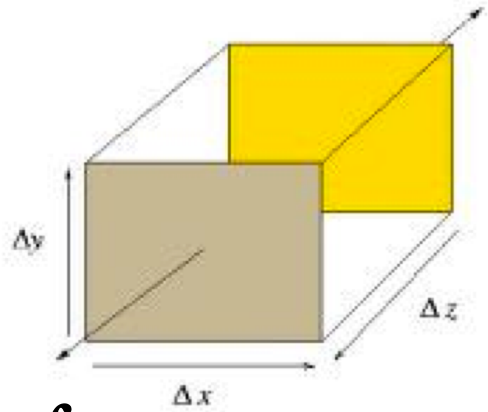
$$(\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{A} = \boldsymbol{\nabla}(\frac{1}{2}A^2) - \boldsymbol{A} \times (\boldsymbol{\nabla} \times \boldsymbol{A})$$

If there are no collisions and no thermal motion, all particles in a fluid move together. The average velocity of the fluid in an element equals the individual particle velocity.

The Plasma as a Fluid

Conservation of matter

The total number of particles can be altered only if there is a net flux of particles across the surface S which bounds the volume V



Particle flux density: $\boldsymbol{J} = n\boldsymbol{v}$

Divergence theorem: $\int_V (\nabla \cdot \boldsymbol{A})\, dV = \oint_s \boldsymbol{A} \cdot d\boldsymbol{S}$

$$\frac{\partial N}{\partial t} = \int_V \frac{\partial n}{\partial t} dV = -\oint_s n\boldsymbol{v} \cdot d\boldsymbol{S} = -\int_V \nabla \cdot (n\boldsymbol{v}) dV \quad \text{for any volume V} \quad \Longrightarrow$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\boldsymbol{v}) = 0$$

Equation of continuity for each specie of the plasma

The Plasma as a Fluid

Equation of state

$$p = Cn^{\gamma}$$

γ is the ratio of the specific heats C_p/C_v $\gamma = \frac{2+N}{N}$

N is the number of degrees of freedom

$$\frac{\nabla p}{p} = \gamma \frac{\nabla n}{n}$$

prove this

isothermal change $\gamma=1$
adiabatic/isotropic 3 degrees of freedom $\gamma=5/3$
adiabatic 1 degree of freedom $\gamma=3$
adiabatic 2 degree of freedom $\gamma=2$

isothermal compression:

$$\nabla p = \nabla(nKT) = KT\nabla n$$

ideal gas: $p = nKT$

$$n = \frac{N}{V}$$ particle density

The Plasma as a Fluid

Convective derivative

In the absence of collisions and thermal motion, the fluid equation could be obtained by multiplying the equation of motion by the density of the species *n*

$$mn\frac{\partial \boldsymbol{\upsilon}}{\partial \mathrm{t}} = qn\left(\boldsymbol{E} + \boldsymbol{\upsilon} \times \boldsymbol{B}\right)$$

In eq. of motion, time derivative is taken at the position of particles (remember the non uniform E - finite Larmor-radius treatment).
But we are keen to have an equation for fluid elements fixed in space, thus need a transformation to variables in a fixed frame. Assume *G(x,t)* to be any variable of fluid.

$$\frac{d\boldsymbol{G}(x,t)}{dt} = \frac{\partial \boldsymbol{G}}{\partial t} + \frac{\partial \boldsymbol{G}}{\partial x}\frac{dx}{dt} = \frac{\partial \boldsymbol{G}}{\partial t} + \upsilon_x \frac{\partial \boldsymbol{G}}{\partial x} \equiv \frac{D\boldsymbol{G}}{Dt}$$

Change at a fixed point

Change of G as the observer moves with the fluid into a region in which G is different

3D:

$$\frac{d\boldsymbol{G}(x,t)}{dt} = \frac{\partial \boldsymbol{G}}{\partial t} + (\boldsymbol{\upsilon} \cdot \boldsymbol{\nabla})\boldsymbol{G}$$

$\boldsymbol{\upsilon} \cdot \boldsymbol{\nabla}$ is a scalar differential operator

The Plasma as a Fluid

If there are no collisions and no thermal motion, all particles in a fluid move together. The average velocity of the fluid in an element equals the individual particle velocity.

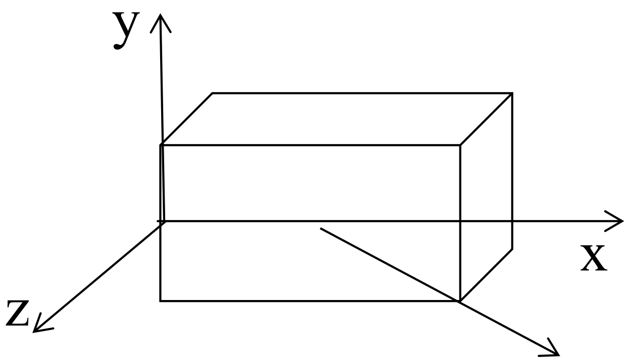
For a plasma **G**=**υ** the fluid velocity:

$$mn\left(\frac{\partial \boldsymbol{\upsilon}}{\partial t} + (\boldsymbol{\upsilon} \cdot \boldsymbol{\nabla})\boldsymbol{\upsilon}\right) = qn\left(\boldsymbol{E} + \boldsymbol{\upsilon} \times \boldsymbol{B}\right)$$

$\frac{\partial \boldsymbol{\upsilon}}{\partial t} \equiv$ Time derivative in a fixed frame

Thermal motions:

If thermal motions are considered, a pressure force has to be included in the equation due to the random motion of the particles in the plasma fluid.



$(x_o, 1/2\delta y, 1/2\delta z)$

Lets consider for simplicity only the x component of the motion.

The Plasma as a Fluid

Ordinary fluid dynamics
Navier – Stokes equation

(1) $\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right) = -\boldsymbol{\nabla} p + \rho\nu\nabla^2\boldsymbol{v} \longleftarrow$ Viscosity term, ν is the kinematic viscosity coefficient

Comparison with the plasma motion equation:

$$(2)\ mn\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right) = qn\left(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\right) - \boldsymbol{\nabla}\cdot\mathbf{P} - \frac{mn(\boldsymbol{v}-\boldsymbol{v}_o)}{\boldsymbol{\tau}}$$

$\rho\nu\nabla^2\boldsymbol{v} \rightarrow$ Viscosity term corresponds to the collisional part in the fluid eq.

Equation (1) describes a collisional fluid with frequent collisions between particles.
Equation 2 was derived without “collision rate” definition **between plasma species**, but eq. (2) indeed can describe the plasma species. Since we used the Maxwellian velocity distribution we implicitly considered collisions.

The Plasma as a Fluid

The complete set of equations

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_q}{\varepsilon_o} = \frac{n_i q_i + n_e q_e}{\varepsilon_o}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \mu_o (\boldsymbol{j} + \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t}) =$$

$$= \mu_o (n_i q_i \boldsymbol{\upsilon}_i + n_e q_e \boldsymbol{\upsilon}_e + \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t})$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$p_j = C_j n_j^{\gamma} \; j = i, e$$

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \boldsymbol{\upsilon}_j) = 0$$

$$\boldsymbol{j} = n_i q_i \boldsymbol{\upsilon}_i + n_e q_e \boldsymbol{\upsilon}_e$$

$$\rho_q = n_i q_i + n_e q_e$$

All charges are included in ρ_q, bound & free

$$m_j n_j (\frac{\partial \boldsymbol{\upsilon}_j}{\partial t} + (\boldsymbol{\upsilon}_j \cdot \nabla) \boldsymbol{\upsilon}_j) = q_j n_j \left(\boldsymbol{E} + \boldsymbol{\upsilon}_j \times \boldsymbol{B} \right) - \nabla \boldsymbol{p}_j$$

j=i,e

Waves in plasmas

Waves in plasmas

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_o} = \frac{n_i q_i + n_e q_e}{\varepsilon_o}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \mu_o (\boldsymbol{j} + \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t}) =$$

$$= \mu_o (n_i q_i \boldsymbol{\upsilon}_i + n_e q_e \boldsymbol{\upsilon}_e + \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t})$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$p_j = C_j n_j^{\gamma} \quad j = i,e$$

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \boldsymbol{\upsilon}_j) = 0$$

$$\boldsymbol{j} = n_i q_i \boldsymbol{\upsilon}_i + n_e q_e \boldsymbol{\upsilon}_e$$

$$\rho_q = n_i q_i + n_e q_e$$

1. Linearisation of plasma equations:

$$s = s_o + s_1$$

s_o is the plasma variable in equilibrium

s_1 is the perturbation of the plasma variable in equilibrium

2. the perturbation is a planar harmonic wave around the point of equilibrium S_o:

$$s_1 = s_{10} e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}$$

3. Solve the linearised equations of the plasma to find the *dispersion relation* $\omega(\boldsymbol{\kappa})$:

4. We ignore the influence of the oscillations on the unperturbed quantities i.e. S_o):

$$m_j n_j (\frac{\partial \boldsymbol{\upsilon}_j}{\partial t} + (\boldsymbol{\upsilon}_j \cdot \nabla) \boldsymbol{\upsilon}_j) = q_j n_j (\boldsymbol{E} + \boldsymbol{\upsilon}_j \times \boldsymbol{B}) - \nabla \boldsymbol{p}_j \qquad j = i,e$$

Plasma - One Fluid Equations - MHD

$$p_j = C_j n_j^{\gamma} \quad j = i, e$$

$$m_j n_j \left(\frac{\partial \boldsymbol{v}_j}{\partial t} + (\boldsymbol{v}_j \cdot \boldsymbol{\nabla})\boldsymbol{v}_j\right) = q_j n_j \left(\boldsymbol{E} + \boldsymbol{v}_j \times \boldsymbol{B}\right) - \boldsymbol{\nabla} \boldsymbol{p}_j$$

$$\frac{\partial n_j}{\partial t} + \boldsymbol{\nabla} \cdot (n_j \boldsymbol{v}_j) = 0$$

Current density: $\boldsymbol{J}(\boldsymbol{r}, t) = e\left(n_i \boldsymbol{v}_i - n_e \boldsymbol{v}_e\right)$

$$\boldsymbol{J} = n_i q_i \boldsymbol{v}_i + n_e q_e \boldsymbol{v}_e$$

$$\rho_q = n_i q_i + n_e q_e$$

Charge density: $\rho_q(\boldsymbol{r}, t) = e\left(n_i - n_e\right)$

For electrically neutral plasma:

$q_e n_e \sim q_i n_i$

Mass density: $\rho_m(\boldsymbol{r}, t) = \left(n_e m_e + n_i m_i\right) \cong n_i m_i$

because $m_e << m_i \quad n_e \cong n_i$

Pressure: $p = p_e + p_i$

For isotropic plasma:

$$p_j = n_j m_j v_j^2 = n_j K_B T_j$$

Mass centre vel. of fluid: $\boldsymbol{v} = \left(m_i n_i \boldsymbol{v}_i + m_e n_e \boldsymbol{v}_e\right) / \rho_m$

MHD Equations

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \boldsymbol{\upsilon}) = 0$$

$$\frac{\partial \rho_q}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$

$$\rho_m \frac{\partial \boldsymbol{\upsilon}}{\partial t} = -\nabla P + \rho_q \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}$$

$$\left(\frac{m_e m_i}{e^2 \rho_m}\right)\left(\frac{\partial \boldsymbol{J}}{\partial t}\right) = \boldsymbol{E} + \boldsymbol{\upsilon} \times \boldsymbol{B} - \frac{m_i}{e\rho_m} \boldsymbol{J} \times \boldsymbol{B} + \frac{m_i}{2e\rho_m} \nabla P - \frac{\boldsymbol{J}}{\sigma}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \mu_o (\boldsymbol{J} + \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t})$$

$$p = Cn^{\gamma}$$

Waves in plasmas

Ideal Magneto-Hydro Dynamics (Ideal MHD)

If we also consider that there is no accumulation of space charge i.e. $\rho_q=0$, the Simplified MHD equations including Maxwell equations are:

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \boldsymbol{\upsilon}) = 0$$

$$\rho_m \frac{\partial \boldsymbol{\upsilon}}{\partial t} = -\nabla P + \boldsymbol{J} \times \boldsymbol{B}$$

$$\boldsymbol{J} = \sigma\left(\boldsymbol{E} + \boldsymbol{\upsilon} \times \boldsymbol{B}\right)$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J}$$

$$p = Cn^{\gamma}$$

$$\nabla \cdot \boldsymbol{E} = 0 \qquad \nabla \cdot \boldsymbol{B} = 0$$

Closed system of equations – Good luck!!

Waves in plasmas

Sound waves in a fluid: ion acoustic waves in a plasma

$$\rho_m\left(\frac{\partial \boldsymbol{\upsilon}}{\partial t}+(\boldsymbol{\upsilon}\cdot\nabla)\boldsymbol{\upsilon}\right)=-\nabla \boldsymbol{p}=-\gamma\frac{p}{\rho_m}\nabla\rho_m$$

Navier-Stokes – no viscosity

$$\frac{\nabla p}{p}=\gamma\frac{\nabla\rho_m}{\rho_m}$$

$$\frac{\partial\rho_m}{\partial t}+\nabla\cdot(\rho_m\boldsymbol{\upsilon})=0$$

Continuity equation

ideal gas: $p=\dfrac{\rho_m KT}{m}$

Linearising about a stationary equilibrium with uniform p_o and ρ_o: (keep only 1st order terms)

$$\rho_m=\rho_{mo}+\rho_{1mo}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$

Oscillation about the density ρ_o of the fluid

$$\boldsymbol{\upsilon}=\upsilon_o+\upsilon_{10}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)},\upsilon_o=0$$

Fluid is immobile and we investigate the vibrations of its velocity

$$-i\omega\rho_1+\rho_o i\boldsymbol{k}\cdot\boldsymbol{\upsilon}_1=0$$

Continuity equation

$$\boldsymbol{k}=k\boldsymbol{x},\ and\ \boldsymbol{\upsilon}=\upsilon\boldsymbol{x}$$

$$-i\omega\rho_o\boldsymbol{\upsilon}_1=-\gamma\frac{p_o}{\rho_o}i\boldsymbol{k}\rho_1$$

Navier-Stokes

Eliminating ρ_1:

Waves in plasmas

$$p = \frac{\rho_{mo} KT}{m}$$

Eliminating ρ_1:

$$-i\omega\rho_{mo}\boldsymbol{\upsilon}_1 = -\gamma \frac{p_o}{\rho_{mo}} i\boldsymbol{k} \frac{\rho_{mo} ik\upsilon_1}{i\omega} \Leftrightarrow \omega^2\upsilon_1 = k^2\gamma \frac{p_o}{\rho_{mo}} \upsilon_1 \Leftrightarrow$$

$$\frac{\omega}{k} = \left(\gamma \frac{p_o}{\rho_{mo}}\right)^{1/2} = \left(\gamma \frac{KT}{m}\right)^{1/2} \equiv c_s$$

Sound waves velocity in a neutral gas. These are pressure waves which propagate through collisions among the air molecules.

In a plasma: Ion acoustic waves

Low frequency oscillations because of the large mass of the ion.

Ion fluid equation:

$$m_i n\left(\frac{\partial \boldsymbol{\upsilon}_i}{\partial t} + (\boldsymbol{\upsilon}_i \cdot \boldsymbol{\nabla})\boldsymbol{\upsilon}_i\right) = en\boldsymbol{E} - \boldsymbol{\nabla} p = -en\boldsymbol{\nabla}\phi - \gamma_i KT_i \boldsymbol{\nabla} n$$

For the electrons we assume m_e=0
Low frequency so we use plasma approximation, so:

$$n_i = n_e = n$$

Linearising (assuming plane waves): $-i\omega m_i n_o \upsilon_{i1} = -eik\phi_1 - \gamma_i KT_i ikn_1$

Waves in plasmas

Ion sound waves in a plasma

The balance of forces on electrons requires:

$$n_e = n = n_o e^{\frac{e\phi_1}{KT_e}} = n_o\left(1 + \frac{e\phi_1}{KT_e} + \ldots\right)$$

Boltzmann relation for electrons

$$n_e = n_{oe} e^{\frac{e\phi}{KT_e}}$$

Since $E_o = 0$, $\varphi_o = 0$:

Electrons are very mobile that their heat conductivity is huge! So we can consider isothermal electrons i.e. $\gamma=1$

The perturbation in the electron and therefore the ion density can be written:

$$n_1 = n_o \frac{e\phi_1}{KT_e}$$

Linearising the ion equation of continuity:

$$\frac{\partial n_1}{\partial t} + \boldsymbol{\nabla}\cdot(n_o \boldsymbol{\upsilon}_{\iota 1} + n_1 \boldsymbol{\upsilon}_{\iota 1}) = 0 \Leftrightarrow \frac{\partial n_1}{\partial t} + n_o \boldsymbol{\nabla}\cdot\boldsymbol{\upsilon}_{\iota 1} + \boldsymbol{\upsilon}_{\iota 1}\cdot\boldsymbol{\nabla} n_o) = 0 \Leftrightarrow$$

$$\Leftrightarrow -i\omega n_1 = -n_o ik\upsilon_{\iota 1}$$

Waves in plasmas

Ion sound waves in a plasma

So the set of equations are:

$$-i\omega m_i n_o \upsilon_{i1} = -eik\phi_1 - \gamma_\iota KT_i ikn_1$$

$$-i\omega n_1 = -n_o ik\upsilon_{\iota 1}$$

$$n_1 = n_o \frac{e\phi_1}{KT_e}$$

Substituting for φ_1 and n_1 in the equation of motion:

$$i\omega m_i n_o \upsilon_{i1} = en_o ik\frac{KT_e}{en_o} + \gamma_\iota KT_i ik\frac{n_o ik\upsilon_{i1}}{i\omega} \Leftrightarrow \omega^2 = k^2\left(\frac{KT_e + \gamma_\iota KT_\iota}{m_i}\right)$$

$$\frac{\omega}{k} = \left(\frac{KT_e + \gamma_\iota KT_\iota}{m_i}\right)^{1/2} \equiv \upsilon_s$$

Ions suffer 1-D compression due to the plane wave we can use $\gamma_\iota=3$.
Electrons are so fast that they have time to equalise their temperature everywhere, so they are isothermal and $\gamma_e=1$.

Dispersion relation of ion acoustic wave – υ_s is the sound speed in a plasma

Waves in plasmas

Electron waves in a plasma

B=0 i.e. non magnetised plasma T_e=T_i=0 i.e. no thermal motion, ions are fixed in space and have uniform distribution, plasma is infinite in space.

Let's display electrons from the uniform ion background: E-field will be generated to restore neutrality.

Equation of motion for electrons:

$$m_e n_e\left(\frac{\partial \boldsymbol{\upsilon}_e}{\partial t} + (\boldsymbol{\upsilon}_e \cdot \boldsymbol{\nabla})\boldsymbol{\upsilon}_e\right) = -en_e\boldsymbol{E}$$

Continuity equation:

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \boldsymbol{\upsilon}_e) = 0$$

$$\varepsilon_o \nabla \cdot \boldsymbol{E} = \varepsilon_o \frac{\partial E}{\partial x} = e(n_i - n_e)$$

We chose 1D for simplicity

Linearisation:

$$n_e = n_o + n_1 \qquad \upsilon_e = \upsilon_o + \upsilon_1 \qquad E = E_o + E_1$$

$$\upsilon_1 = \upsilon_{10} e^{i(kx-\omega t)} \qquad n_1 = n_{10} e^{i(kx-\omega t)} \qquad E = E_0 e^{i(kx-\omega t)}$$

Waves in plasmas

Electron waves in a plasma

Initial conditions (uniform neutral plasma): In equilibrium: $n_{io}=n_{eo}$ and $n_{i1}=0$

$$\nabla n_o = \upsilon_o = E_o = 0 \qquad \frac{\partial}{\partial t} n_o = \frac{\partial}{\partial t}\upsilon_o = \frac{\partial}{\partial t}E_o = 0$$

$$m_e n_e\left(\frac{\partial \boldsymbol{\upsilon}_e}{\partial t} + (\boldsymbol{\upsilon}_e \cdot \boldsymbol{\nabla})\boldsymbol{\upsilon}_e\right) = -en_e\boldsymbol{E} \quad \Rightarrow \quad m_e\left(\frac{\partial \boldsymbol{\upsilon}_1}{\partial t} + (\boldsymbol{\upsilon}_1 \cdot \boldsymbol{\nabla})\boldsymbol{\upsilon}_1\right) = -e\boldsymbol{E}$$

0 because quadratic

$$\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e\boldsymbol{\upsilon}_e) = 0 \quad \Rightarrow \quad \frac{\partial n_1}{\partial t} + \nabla\cdot(n_o\boldsymbol{\upsilon}_1 + n_1\boldsymbol{\upsilon}_1) = 0$$

0

$$\Rightarrow \quad \frac{\partial n_1}{\partial t} + n_o\nabla\cdot\boldsymbol{\upsilon}_1 + \boldsymbol{\upsilon}_1\cdot\nabla n_o = 0$$

0

From Poisson's equation:

$$\varepsilon_o\frac{\partial E_1}{\partial x} = -en_1$$

Since $n_{io}= n_{eo}$ and $n_{i1}=0$ by the assumption of fixed ions

Waves in plasmas

Electron waves in a plasma

So there are three equations and three unknowns, linearising:

$$-i\omega m_e \upsilon_1 = -eE_1$$

$$-i\omega n_1 = -n_o ik\upsilon_1$$

$$ik\varepsilon_o E_1 = -en_1$$

Eliminating n_1 and E_1

$$-i\omega m_e \upsilon_1 = -e\frac{-e}{ik\varepsilon_o}\frac{-n_o ik\upsilon_1}{-i\omega} = -i\frac{n_o e^2}{\varepsilon_o \omega}\upsilon_1$$

$$\Rightarrow \omega^2 = \frac{n_o e^2}{m_e \varepsilon_o} \text{ rad/sec}$$

Characteristic plasma frequency

$$\omega_{pe}^2 = \frac{n_o e^2}{m_e \varepsilon_o}$$

Plasma frequency

$$\frac{\omega_{pe}}{2\pi} = f_{pe} \sim 9\sqrt{n_e}$$

ω does not depend on k
Group velocity is zero
No propagation

$f_{pe} \sim 9GHz!$ for $n_o = 10^{18} m^{-3}$ $\qquad f_{ce} \sim 28GHz / Tesla$

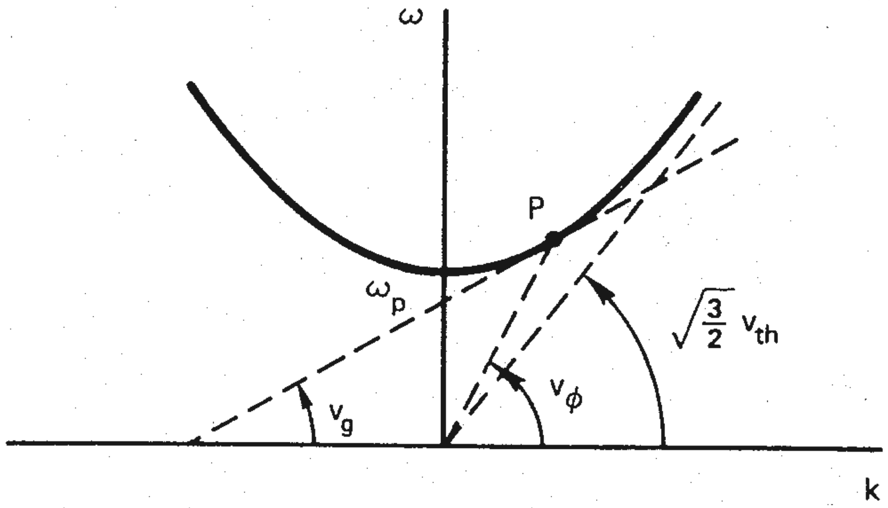
$f_{pe} \sim f_{ce}$ for $B \sim 0.32 Tesla$ and $n \sim 10^{18} m^{-3}$

Waves in plasmas

Thermal electron waves in a plasma

$$\nabla p_e = 3KT_e \nabla n_e$$

$$\omega^2 = \omega_p^2 + \frac{3}{2} k^2 \upsilon_{th}^2$$



Waves in plasmas

Magnetised plasmas

The existence of an external $\boldsymbol{B}_o$ adds one more parameter to the cases we have studied i.e.:
Electrostatic waves ($\boldsymbol{B}_1$=0) of low or high frequencies,
Electromagnetic waves ($\boldsymbol{B}_1 \neq 0$) of low or high frequencies,

The new parameter is now whether the propagation is:

1) Perpendicular to $\boldsymbol{B}_o$: $\boldsymbol{k} \cdot \boldsymbol{B}_o = 0$
2) Parallel to $\boldsymbol{B}_o$: $\boldsymbol{k} \times \boldsymbol{B}_o = 0$

Note that waves travelling parallel to the perturbation $\boldsymbol{E}_1$ are longitudinal i.e. $\boldsymbol{k} \times \boldsymbol{E}_1 = 0$ while waves travelling perpendicular to the perturbation $\boldsymbol{E}_1$ are transverse i.e. $\boldsymbol{k} \cdot \boldsymbol{E}_1 = 0$

Home work: Show that all electrostatic waves are longitudinal and vise versa and that all transverse waves are electromagnetic but the opposite is not true.

Waves in plasmas

High Frequency Electrostatic waves propagating transverse to external B field
Upper Hybrid Frequency (UHF)

We will apply the same approaches as in non magnetised plasmas, "cold plasma", T_e=T_i=0 i.e. no thermal motion (thus P_e=P_i=0) ions are fixed in space and have uniform distribution, plasma is infinite in space. Since the waves have high frequency, ions can be considered immobile because of their large mass.

Since electrostatic waves are longitudinal, $\boldsymbol{k} \times \boldsymbol{E}_1 = 0$ and we also consider that $\boldsymbol{k} \cdot \boldsymbol{B}_o = 0$

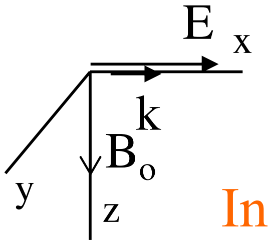
To identify the dispersion relation the following equations are needed:

$$m_e n_e \left(\frac{\partial \boldsymbol{\upsilon}_e}{\partial t} + (\boldsymbol{\upsilon}_e \cdot \boldsymbol{\nabla})\boldsymbol{\upsilon}_e\right) = -e n_e \left[\boldsymbol{E}_1 + \boldsymbol{\upsilon}_e \times \boldsymbol{B}_o\right] \qquad \frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \boldsymbol{\upsilon}_e) = 0$$

$$\varepsilon_o \boldsymbol{\nabla} \cdot \boldsymbol{E}_1 = e(n_o - n_e) \qquad n_{io} = n_{eo} = n_o$$

Waves in plasmas

High Frequency Electrostatic waves propagating transverse to external B field
Upper Hybrid Frequency (UHF)



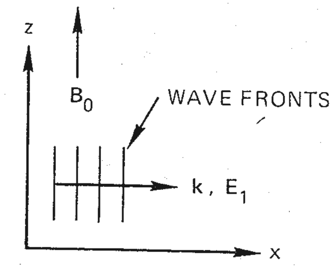
Lets consider that $\mathbf{E}_1 = E_1 \mathbf{x}$, $\mathbf{k} = k\mathbf{x}$, $\mathbf{B}_o = B_o \mathbf{z}$

In order for the eq. of motion to be obeyed, υ_e should lye on the (x,y) plane

Linearising the three equations (motion, continuity, Poisson) and keeping first order terms:

$$-i\omega m_e \upsilon_{e1x} = -eE_1 - e\upsilon_{e1y} B_0 \qquad -i\omega m_e \upsilon_{e1y} = e\upsilon_{e1x} B_0$$

$$-i\omega n_{e1} + ikn_o \upsilon_{e1x} = 0 \qquad ik\varepsilon_o E_1 = -en_{e1}$$



Geometry of a longitudinal plane wave propagating at right angles to B_0.

From the first two (eq. of motion):

$$\upsilon_{e1x} = \frac{eE_1}{im_e\omega\left(1 - \dfrac{\Omega_e^2}{\omega_e^2}\right)} \qquad \Omega_e \equiv \frac{eB_o}{m_e}$$

Substituting in the third (continuity) and using the fourth (Poisson):

$$\left(-\frac{\omega^2}{\omega_e^2} + 1 + \frac{\Omega_e^2}{\omega_e^2}\right)E_1 = 0 \Leftrightarrow \omega^2 = \omega_e^2 + \Omega_e^2 \equiv \omega_{UH}^2$$

Waves in plasmas

High Frequency Electrostatic waves propagating transverse to external B field
Upper Hybrid Frequency (UHF)

The upper hybrid frequency (πάνω υβριδική συχνότητα) is equivalent to the plasma Frequency for waves travelling transverse to the external magnetic field.

The electrons perform an additional cyclotron oscillation due to the presence of the external magnetic field.

Group velocity is zero as long as thermal motion is neglected.

Electrostatic electron wave along B_o are the usual plasma oscillations with $\omega=\omega_p$

Home work 1: Why in the above analysis we ignored the last three Maxwell's equations (Gauss's law for magnetism, Ampere's law, Faraday's law) and we used only the Poisson's equation.

Waves in plasmas

Low Frequency Electrostatic waves propagating transverse to external B field
Lower Hybrid Frequency (LHF)

Since electrostatic waves are investigated, only Poisson's equation could be used From the Maxwell's equations, but since frequency is low ions can follow the vibrations of electrons and $n_{e1}=n_{i1}$. So we have to use the equations of motion and continuity for both ions and electrons, thus the equations which describe the propagation of such waves are:

$$m_e n_e\left(\frac{\partial \boldsymbol{\upsilon}_e}{\partial t} + (\boldsymbol{\upsilon}_e \cdot \nabla)\boldsymbol{\upsilon}_e\right) = -en_e\left[\boldsymbol{E}_1 + \boldsymbol{\upsilon}_e \times \boldsymbol{B}_o\right] \qquad \frac{\partial n_e}{\partial t} + \nabla\cdot(n_e\boldsymbol{\upsilon}_e) = 0$$

$$m_i n_i\left(\frac{\partial \boldsymbol{\upsilon}_i}{\partial t} + (\boldsymbol{\upsilon}_i \cdot \nabla)\boldsymbol{\upsilon}_i\right) = -en_i\left[\boldsymbol{E}_1 + \boldsymbol{\upsilon}_i \times \boldsymbol{B}_o\right] \qquad \frac{\partial n_i}{\partial t} + \nabla\cdot(n_i\boldsymbol{\upsilon}_i) = 0$$

$$n_{io} = n_{eo} = n_o$$

After linearisation:

Waves in plasmas

Low Frequency Electrostatic waves propagating transverse to external B field
Lower Hybrid Frequency (LHF)

$$m_e n_o \frac{\partial \boldsymbol{\upsilon}_e}{\partial t} = -\gamma_e k_B T_e \nabla n_{e1} - e n_o \left[\boldsymbol{E}_1 + \boldsymbol{\upsilon}_e \times \boldsymbol{B}_o \right] \qquad \frac{\partial n_{e1}}{\partial t} + n_o \nabla \cdot \boldsymbol{\upsilon}_e = 0$$

$$m_i n_o \frac{\partial \boldsymbol{\upsilon}_i}{\partial t} = -\gamma_i k_B T_i \nabla n_{e1} + e n_o \left[\boldsymbol{E}_1 + \boldsymbol{\upsilon}_i \times \boldsymbol{B}_o \right] \qquad \frac{\partial n_{e1}}{\partial t} + n_o \nabla \cdot \boldsymbol{\upsilon}_i = 0$$

$$\nabla P_j = \gamma_j k_B T_j \nabla n_j$$

where we have used that $n_{e1}=n_{i1}$

$$solutions \sim e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$

As previously $\mathbf{B}_o=B_o\mathbf{z}$, and the waves propagate on the (x,z) plane parallel with the perturbation of the **E** field, $\mathbf{k}//\mathbf{E}_1$

Add by parts:

$$-i\omega n_o(m_e\boldsymbol{\upsilon}_e + m_i\boldsymbol{\upsilon}_i) = -i\boldsymbol{k}n_{e1}(\gamma_e k_B T_e + \gamma_i k_B T_i) + e n_o(\boldsymbol{\upsilon}_i - \boldsymbol{\upsilon}_e)\times \boldsymbol{B}_o$$

Waves in plasmas

Low Frequency Electrostatic waves propagating transverse to external B field
Lower Hybrid Frequency (LHF)

From the continuity equations : $\boldsymbol{k}\cdot\boldsymbol{\upsilon}_{e1} = \boldsymbol{k}\cdot\boldsymbol{\upsilon}_{i1} = \dfrac{\omega n_{e1}}{n_o}$ So equation of motion becomes:

$$-i\omega^2 n_o(m_e + m_i)\frac{n_{e1}}{n_o} = -ik^2 n_{e1}(\gamma_e k_B T_e + \gamma_i k_B T_i) + en_o\boldsymbol{k}\cdot[(\boldsymbol{\upsilon}_i - \boldsymbol{\upsilon}_e)\times\boldsymbol{B}_o] \quad (1)$$

$$\boldsymbol{k}\cdot[(\boldsymbol{\upsilon}_i - \boldsymbol{\upsilon}_e)\times\boldsymbol{B}_o] = k_x[(\upsilon_{iy} - \upsilon_{ey})B_o]$$

But to find the dispersion relation from this equation we need to express the velocities as functions of the density.

For this reason we externally multiply the electron equation of motion (two slides before) with **k** and we take:

$$-i\omega m_e n_o(\boldsymbol{k}\times\boldsymbol{\upsilon}_e) = -en_o\boldsymbol{k}\times(\boldsymbol{\upsilon}_e\times\boldsymbol{B}_o) \quad \left\{ \begin{array}{l} \upsilon_{ex} = -\dfrac{i\omega}{\Omega_e}\upsilon_{ey} \\[2ex] \upsilon_{ez} = +i\upsilon_{ey}\dfrac{k_z}{k_x}\dfrac{\Omega_e}{\omega}\left(1 - \dfrac{\omega^2}{\Omega_e^2}\right) \end{array} \right.$$

Waves in plasmas

Low Frequency Electrostatic waves propagating transverse to external B field
Lower Hybrid Frequency (LHF)

Electron continuity equations becomes:

$$k_x \upsilon_{ex} + k_z \upsilon_{ez} = \omega \frac{n_{e1}}{n_o} \Leftrightarrow \omega \frac{n_{e1}}{n_o} = \left[k_x \left(-\frac{i\omega}{\Omega_e} \right) + k_z \left(\frac{+i\Omega_e}{\omega} \right) \frac{k_z}{k_x} \left(1 - \frac{\omega^2}{\Omega_e^2} \right) \right] \upsilon_{ey}$$

By substituting υ_{ey} in the previous relations for υ_{ex}, υ_{ez} we have written υ_{ex}, υ_{ez} as Functions of n_{e1}. Substituting in eq (1) of the previous slide and considering $m_e \ll m_i$, we obtain the dispersion relation:

$$1 - \frac{k^2 c_s^2}{\omega^2} + \frac{\Omega_i}{\omega} \left[\frac{1}{-\frac{\omega}{\Omega_e} + \frac{\Omega_e}{\omega} \frac{k_z^2}{k_x^2} \left(1 - \frac{\omega^2}{\Omega_e^2} \right)} - \frac{1}{\frac{\omega}{\Omega_i} - \frac{\Omega_i}{\omega} \frac{k_z^2}{k_x^2} \left(1 - \frac{\omega^2}{\Omega_i^2} \right)} \right] = 0$$

$$c_s^2 = \frac{\gamma_e k_B T_e + \gamma_i k_B T_i}{m_i}$$

General dispersion relation

Waves in plasmas

Low Frequency Electrostatic waves propagating transverse to external B field
Lower Hybrid Frequency (LHF)

If for instance k_x=0 ($k//B_o$) we obtain : $\omega^2 = c_s^2 k_z^2$ as we have previously calculated for B_o=0 (ιοντακουστικές ταλαντώσεις).

If k_z=0 the dispersion relation is
$$1 - \frac{k_x^2 c_s^2}{\omega^2} - \frac{\Omega_i \Omega_e}{\omega^2} - \frac{\Omega_i^2}{\omega^2} = 0$$

Because $\Omega_i^2 << |\Omega_i \Omega_e|$ we have: $\omega^2 = c_s^2 k_x^2 + \Omega_e \Omega_i$

When $k_x \rightarrow 0$ we have: $\omega_{LH}^2 = \Omega_i \Omega_e$ This is the Lower Hybrid Frequency – κάτω υβριδική συχνότητα

Waves in plasmas

High Frequency Electromagnetic waves propagating transverse to external B field

Perpendicular propagation $\boldsymbol{k} \perp \boldsymbol{B}_o$, if we take transverse waves with $\boldsymbol{k} \perp \boldsymbol{E}_1$ there are two choices, 1) $\boldsymbol{E}_1$ can be parallel to $\boldsymbol{B}_o$ and 2) $\boldsymbol{E}_1$ can be perpendicular to $\boldsymbol{B}_o$

Ordinary wave $\mathbf{k} \perp \mathbf{B_o}, \mathbf{E} \parallel \mathbf{B_o}$

For $\mathbf{E}_1$ parallel to $\mathbf{B}_o$ we can take $\mathbf{B}_o = B_o\mathbf{z}$, $\mathbf{E}_1 = E_1\mathbf{z}$, $\mathbf{k} = k\mathbf{x}$
The wave equation is (as in the case for EM waves with $\mathbf{B}_o=0$)

$$\left(\omega^2 - c^2k^2\right)\boldsymbol{E}_1 = \frac{-i\omega \boldsymbol{j}_1}{\varepsilon_o} = \frac{in_o e\omega \boldsymbol{\upsilon}_{e1}}{\varepsilon_o} \quad \boldsymbol{j}_1 = -en_{oe}\boldsymbol{\upsilon}_{e1} \quad n_{oe} = n_o$$

Because $\mathbf{E}_1 = E_1\mathbf{z}$, we need only the υ_{ez} component which is given by the particle equation of motion

$$m_e \frac{\partial \upsilon_{ez}}{\partial t} = -eE_z$$

Since everything is the same as the equation with B=0 the dispersion relation is:

remember

$$\boldsymbol{\upsilon}_{e1} // \boldsymbol{E}_1 \Rightarrow \boldsymbol{\upsilon}_{e1} \times \boldsymbol{B}_o = 0$$

$$\omega^2 = \omega_{pe}^2 + c^2k^2$$

So the O-wave does not feel the existence of the external B_o

Waves in plasmas

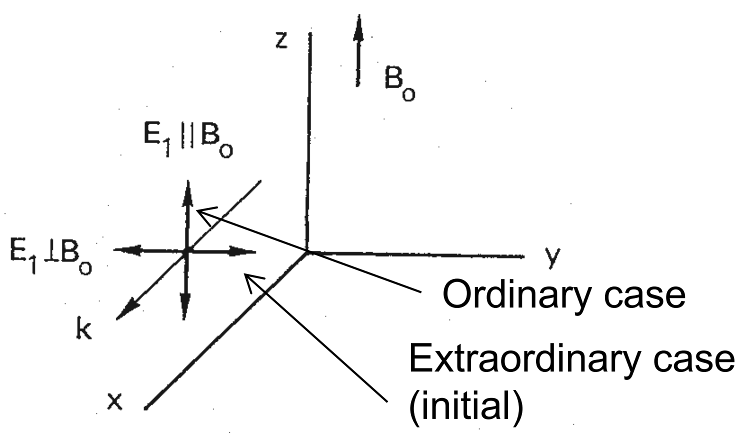
High Frequency Electromagnetic waves propagating transverse to external B field

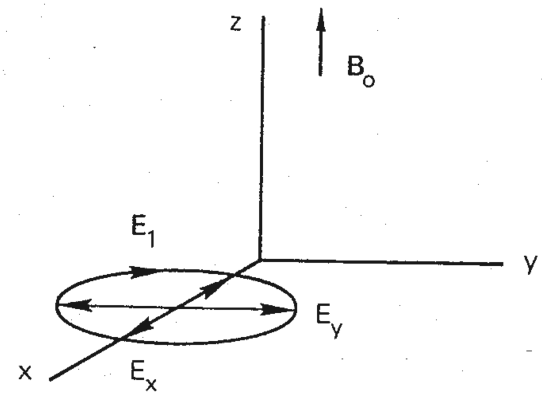
Perpendicular propagation $\boldsymbol{k} \perp \boldsymbol{B}_o$, transverse waves $\boldsymbol{k} \perp \boldsymbol{E}_1$, $\boldsymbol{E}_1$ perpendicular to $\boldsymbol{B}_\mathrm{o}$

Extra-ordinary wave $\mathbf{k} \perp \mathbf{B}_\mathbf{o}, \mathbf{E} \perp \mathbf{B}_\mathbf{o}$

When $\boldsymbol{E}_1$ is perpendicular to $\boldsymbol{B}_\mathrm{o}$, the electron motion will be affected by B_o since $\boldsymbol{v}\mathrm{x}\boldsymbol{B}_\mathrm{o}$ force is not zero as for the previous case (ordinary wave)

To treat this case we can take $\mathbf{B}_\mathrm{o}=B_\mathrm{o}\mathbf{z}$,, $\mathbf{k}=k\mathbf{x}$. However, for the $\mathbf{E}_1$ someone should allow for the development of one more component along **k**, so even if we start with $\mathbf{E}_1=E_1\mathbf{y}$ the $\mathbf{E}_1$ for generality can be $\mathbf{E}_1=E_x\mathbf{x}+E_y\mathbf{y}$ (Actually, it turns out that the wave will become elliptically polarised instead of plane (linearly) polarised, so it will become **partly longidudinal** and **partly transverse**.





Waves in plasmas

High Frequency Electromagnetic waves propagating transverse to external B field

The system of equations is:

$$m_e n_e\left(\frac{\partial \boldsymbol{\upsilon}_e}{\partial t} + (\boldsymbol{\upsilon}_e \cdot \nabla)\boldsymbol{\upsilon}_e\right) = -en_e\left[\boldsymbol{E} + \boldsymbol{\upsilon}_e \times \boldsymbol{B}\right] \qquad \frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \boldsymbol{\upsilon}_e) = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \times \boldsymbol{B} = \mu_o\left(\boldsymbol{j} + \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t}\right) \qquad \boldsymbol{j} = -en_e\boldsymbol{\upsilon}_e$$

The linearised system of equations is (linearised as in previous cases):

$$m_e n_o \frac{\partial \boldsymbol{\upsilon}_{e1}}{\partial t} = -en_o\boldsymbol{E}_1 - n_o\boldsymbol{\upsilon}_{e1} \times \boldsymbol{B}_o \qquad \nabla \times \boldsymbol{E}_1 = -\frac{\partial \boldsymbol{B}_1}{\partial t}$$

$$\nabla \times \boldsymbol{B}_1 = \mu_o\left(\boldsymbol{j} + \varepsilon_o \frac{\partial \boldsymbol{E}_1}{\partial t}\right) \qquad \boldsymbol{j} = -en_o\boldsymbol{\upsilon}_{e1}$$

$$n_e = n_{eo} + n_{e1} \qquad \upsilon_e = \upsilon_{eo} + \upsilon_{e1} \qquad E = E_1 \qquad B = B_o + B_1$$

$$n_{e1} = n_{e10}e^{i(kx-\omega t)} \qquad \upsilon_{e1} = \upsilon_{e10}e^{i(kx-\omega t)} \qquad E_1 = E_{01}e^{i(kx-\omega t)} \qquad B_1 = B_{01}e^{i(kx-\omega t)}$$

$$n_{eo} = n_o$$

$$\upsilon_{eo} = 0$$

Waves in plasmas

High Frequency Electromagnetic waves propagating transverse to external B field

The three linearised equations become:

$$-i\omega m_e \upsilon_x = -eE_x - e\upsilon_y B_o$$

$$-i\omega m_e \upsilon_y = -eE_y - e\upsilon_x B_o$$

$$ikE_y = i\omega B_1$$

$$-ikB_1 = -en_o\upsilon_y - i\omega E_y$$

$$0 = -en_o\upsilon_x - i\omega E_x$$

From the last three:

$$\upsilon_x = \frac{e}{m\omega}\left(-iE_x - \frac{\omega_c}{\omega}E_y\right)\left(\frac{1}{1-\frac{\omega_c^2}{\omega^2}}\right)$$

$$\upsilon_y = \frac{e}{m\omega}\left(-iE_y - \frac{\omega_c}{\omega}E_x\right)\left(\frac{1}{1-\frac{\omega_c^2}{\omega^2}}\right)$$

Replacing into the first two and taking into account that:

$$\omega_{pe}^2 = \frac{n_e e^2}{\varepsilon_o m_e}$$

Waves in plasmas

High Frequency Electromagnetic waves propagating transverse to external B field

$$\begin{bmatrix} \omega^2\left(1-\frac{\omega_c^2}{\omega^2}\right)-\omega_{pe}^2 & i\frac{\omega_{pe}^2\omega_c}{\omega} \\ \left(\omega^2-c^2k^2\right)\left(1-\frac{\omega_c^2}{\omega^2}\right)-\omega_{pe}^2 & -i\frac{\omega_{pe}^2\omega_c}{\omega} \end{bmatrix}\begin{bmatrix} E_x \\ E_y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For E_x, E_y, other than zero this system has a solution only if the determinant is zero

Taking into account that: $\omega_{pe}^2+\omega_c^2=\omega_{UH}^2=\omega_h^2$

$$\left(\omega^2-\omega_h^2\right)\left[\omega^2-\omega_h^2-c^2k^2\left(1-\frac{\omega_c^2}{\omega^2}\right)\right]=\left(\frac{\omega_{pe}^2\omega_c}{\omega}\right)^2$$

$$\frac{c^2k^2}{\omega^2}=\frac{\omega^2-\omega_h^2-\left[\frac{\left(\omega_{pe}^2\omega_c\right)^2}{\omega^2\left(\omega^2-\omega_h^2\right)}\right]}{\omega^2-\omega_c^2}$$

Replacing back the $(\omega_h)^2$ and multiplying through with $\omega^2-\omega_h^2$ we find:

Waves in plasmas

High Frequency Electromagnetic waves propagating transverse to external B field

$$\frac{c^2k^2}{\omega^2} = \frac{c^2}{v_\varphi^2} = 1 - \frac{\omega_{pe}^2}{\omega^2}\frac{\omega^2 - \omega_{pe}^2}{\omega^2 - \omega_h^2}$$

Dispersion relation for the extraordinary wave (X-wave

Cutoff and Resonance frequencies for the X-wave

For $k \rightarrow \infty$ $\omega \rightarrow \omega_h$ so that a **resonance** occurs at a point in the plasma where:

$$\omega_h^2 = \omega_{pe}^2 + \omega_c^2$$

This is the dispersion relation for electrostatic waves propagating across $\mathbf{B}_o$. As the wave approaches the resonant point both its phase velocity and its group velocity approach zero and the energy is converted into upper hybrid oscillations. The X-wave is partly electromagnetic and partly electrostatic, so at resonance this wave losses its electromagnetic character and becomes an electrostatic oscillation.

Waves in plasmas

High Frequency Electromagnetic waves propagating transverse to external B field

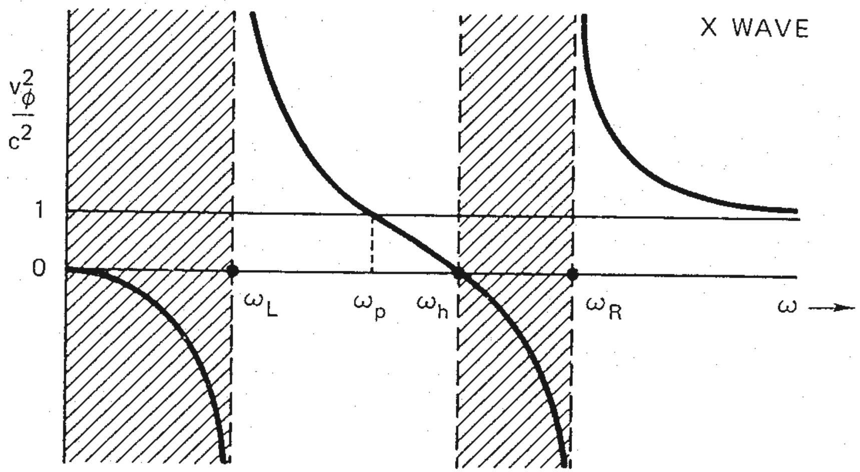
The cutoffs of the X-wave are found by setting k equal to zero at the dispersion relation. After some easy algebra we conclude to the following equation:

$$\left(1-\frac{\omega_{pe}^2}{\omega^2}\right)^2=\frac{\omega_c^2}{\omega^2}\Leftrightarrow 1-\frac{\omega_{pe}^2}{\omega^2}=\pm\frac{\omega_c}{\omega}\Leftrightarrow \omega^2\mp\omega\omega_c-\omega_{pe}^2=0\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}\omega_R=\frac{1}{2}\omega_c+\left(\omega_{pe}^2+\frac{1}{4}\omega_c^2\right)^{1/2}\\ \omega_L=-\frac{1}{2}\omega_c+\left(\omega_{pe}^2+\frac{1}{4}\omega_c^2\right)^{1/2}\end{cases}$$

Waves in plasmas

High Frequency Electromagnetic waves propagating transverse to external B field



- ✓ From the dispersion relation it is clear that at $\omega=\omega_{pe}$ the wave propagates at the velocity of light.
- ✓ For $\omega<\omega_L$ there is another region of non propagation.

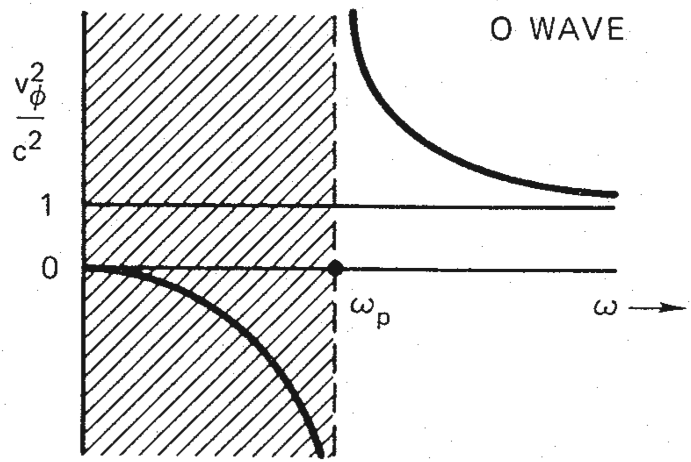
The cutoff and resonance frequencies divide the dispersion diagram into regions of propagation and non propagation.

- ✓ At large ω (or low density) the phase velocity approaches the velocity of light.
- ✓ As the wave travels further the phase velocity increases until the right hand cutoff $\omega=\omega_R$ is met. There the phase velocity becomes infinite.
- ✓ Between $\omega=\omega_R$ and $\omega=\omega_h$ the $(\upsilon_\varphi)^2$ is negative and propagation is prohibited.
- ✓ At $\omega=\omega_h$ there is a resonance and υ_φ goes to zero.
- ✓ Between $\omega=\omega_h$ and $\omega=\omega_L$ propagation is again allowed. In this region the wave travels either faster or slower than c depending on whether ω is smaller or larger than ω_{pe}.

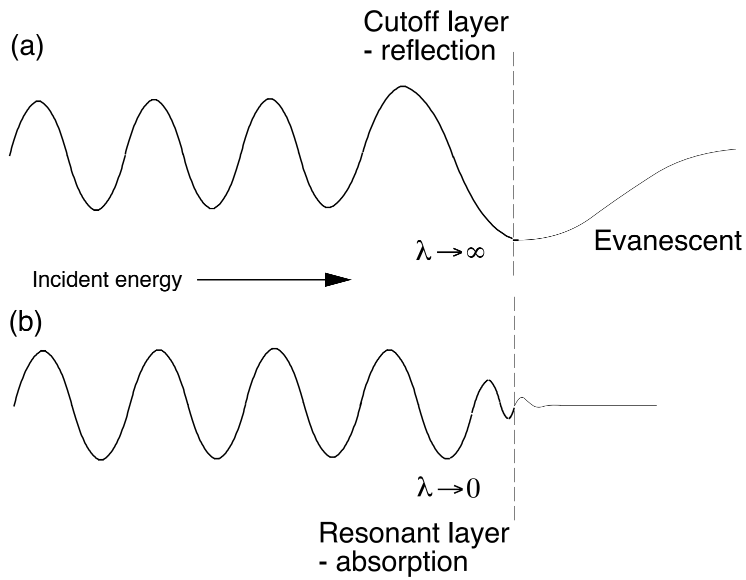
Waves in plasmas

High Frequency Electromagnetic waves propagating transverse to external B field

For the O-wave:



- ✓ Only one cutoff
- ✓ No resonance.



Waves in plasmas

High Frequency Electromagnetic waves propagating parallel to external B field

Let's consider that **k** lies along **z** axis and allow $\mathbf{E}_1$ to have both transverse components (general case)

remember

$$\omega_c \equiv \Omega$$

Wave equation: $\nabla \times (\nabla \times \boldsymbol{E}_1) = \nabla(\nabla \cdot \boldsymbol{E}_1) - \nabla^2 \boldsymbol{E}_1 = -\nabla \times \frac{\partial}{\partial t}\boldsymbol{B}_1$

$$\left.\begin{aligned} (\omega^2 - c^2k^2)E_x &= \frac{\omega_p^2}{1-\frac{\omega_c^2}{\omega^2}}\left(E_x - \frac{i\omega_c}{\omega}E_y\right) \\ (\omega^2 - c^2k^2)E_y &= \frac{\omega_p^2}{1-\frac{\omega_c^2}{\omega^2}}\left(E_y + \frac{i\omega_c}{\omega}E_x\right) \end{aligned}\right\} \quad \delta \equiv \frac{\omega_p^2}{1-\frac{\omega_c^2}{\omega^2}} \quad \Leftrightarrow \quad \left\{\begin{aligned} (\omega^2 - c^2k^2 - \delta)E_x + i\delta\frac{\omega_c}{\omega}E_y &= 0 \\ (\omega^2 - c^2k^2 - \delta)E_y - i\delta\frac{\omega_c}{\omega}E_x &= 0 \end{aligned}\right.$$

Setting determinant equal to zero:

$$(\omega^2 - c^2k^2 - \delta)^2 = \left(\frac{\delta\omega_c}{\omega}\right)^2 = 0 \quad \Leftrightarrow \quad \omega^2 - c^2k^2 - \delta = \pm\frac{\delta\omega_c}{\omega}$$

Waves in plasmas

High Frequency Electromagnetic waves propagating parallel to external B field

...***k*** lies along ***z*** axis and allow $\boldsymbol{E}_1$ to have both transverse components (general case)

$$\omega^2 - c^2k^2 - \delta = \pm\frac{\delta\omega_c}{\omega} \Leftrightarrow \omega^2 - c^2k^2 = \delta\left(1 \pm \frac{\omega_c}{\omega}\right) = \frac{\omega_p^2}{1-\frac{\omega_c^2}{\omega^2}}\left(1 \pm \frac{\omega_c}{\omega}\right) = \omega_p^2\frac{1 \pm \frac{\omega_c}{\omega}}{\left[1+\frac{\omega_c}{\omega}\right]\left[1-\frac{\omega_c}{\omega}\right]} = \frac{\omega_p^2}{1 \mp \frac{\omega_c}{\omega}}$$

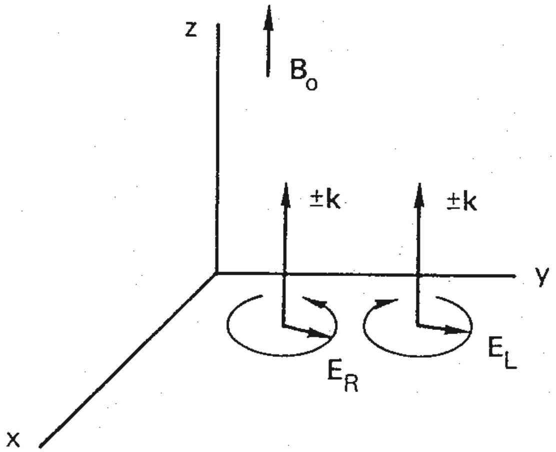
The $\mp$ sign shows that two solutions of two different waves that can propagate along B_o exist. Their dispersion relations are:

$$\eta^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2\left(1-\frac{\omega_c}{\omega}\right)}$$

(-) R-wave

$$\eta^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2\left(1+\frac{\omega_c}{\omega}\right)}$$

(+) L-wave



remember

$$\eta \equiv \frac{c}{v_\phi} = \frac{ck}{\omega}$$

Waves in plasmas

High Frequency Electromagnetic waves propagating parallel to external B field

(-) R-wave Right hand circular polarisation

(+) L-wave Left hand circular polarisation

Since these equations depend only on k^2 the direction of rotation of the ***E*** vector is independent of the sign of ***k*** thus the polarisation is the same for waves propagating in the opposite direction.

Cutoff and Resonance frequencies for R and L waves

For the **R wave** $k \rightarrow \infty$ at $\omega=\omega_c$ the wave is therefore in **resonance** with the cyclotron motion of the electrons. The direction of rotation of the polarisation plane is the same as the direction of the gyration of electrons so the wave loses its energy in continuou-sly accelerating the electrons and therefore cannot propagate.

The **L wave** does **not** have a cyclotron resonance with the electrons because it rotates in the opposite direction. Actually as seen from its dispersion relation the L wave does not have a resonance for positive ω (in some considerations ω can take negative values). (If we had included ions motion, the L wave would have a resonance at $\omega=\omega_{ci}$)

Waves in plasmas

High Frequency Electromagnetic waves propagating parallel to external B field

Remember that in our convection ω is always positive and waves propagating in the – direction (i.e. –x) are described by negative *k*.

Cutoffs: For $k \to 0$

$$\omega_R = \frac{1}{2}\omega_c + \left(\omega_{pe}^2 + \frac{1}{4}\omega_c^2\right)^{1/2}$$

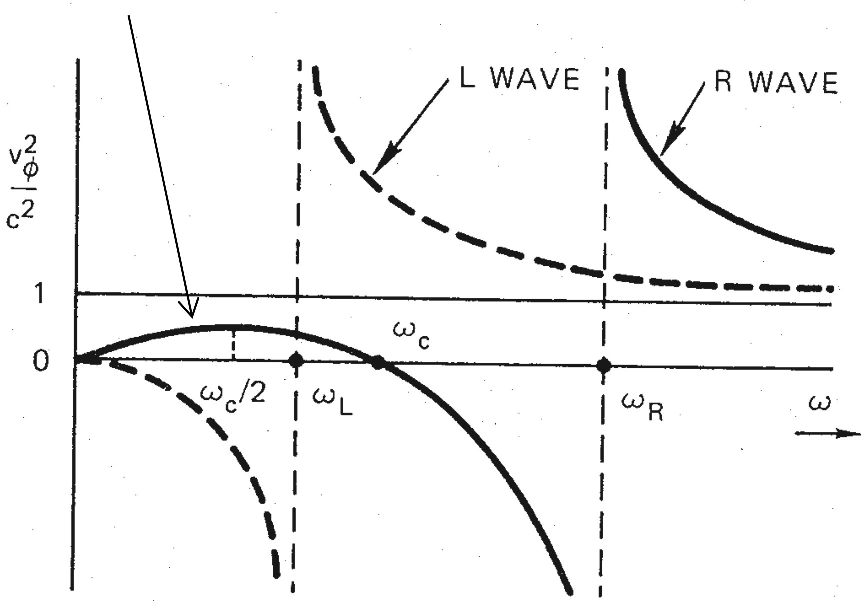
$$\omega_L = -\frac{1}{2}\omega_c + \left(\omega_{pe}^2 + \frac{1}{4}\omega_c^2\right)^{1/2}$$

- ✓ Same equations as for the cutoffs of the X-wave
- ✓ The R-wave (-) has the higher cutoff frequency ω_R while the L-wave (+) has the lower cutoff frequency

Waves in plasmas

High Frequency Electromagnetic waves propagating parallel to external B field

Whistler mode



The dispersion diagram for the R and L waves. Regions of non propagation are for $\upsilon^2_\varphi/c^2 < 0$

- ✓ The L wave (+) has a stop band at low frequencies and it behaves like the O-wave except that the cutoff occurs at ω_L instead of ω_p
- ✓ The R-wave (-) has a stop band between ω_R and ω_c, but there is a second band of propagation with $\upsilon_\varphi < c$ bellow ω_c. The wave in this low frequency region is called the *"whistler mode"* (σφυριχτά) and is of extreme importance for ionospheric phenomena

Waves in plasmas

High Frequency Electromagnetic waves propagating parallel to external B field

Summary

The principal electromagnetic waves propagating along B_o are a Right-hand (R) and a Left-hand (L) circularly polarised wave, the principal waves propagating across B_o are a plane-polarised wave (O-wave) and an elliptically polarised wave (X-wave)

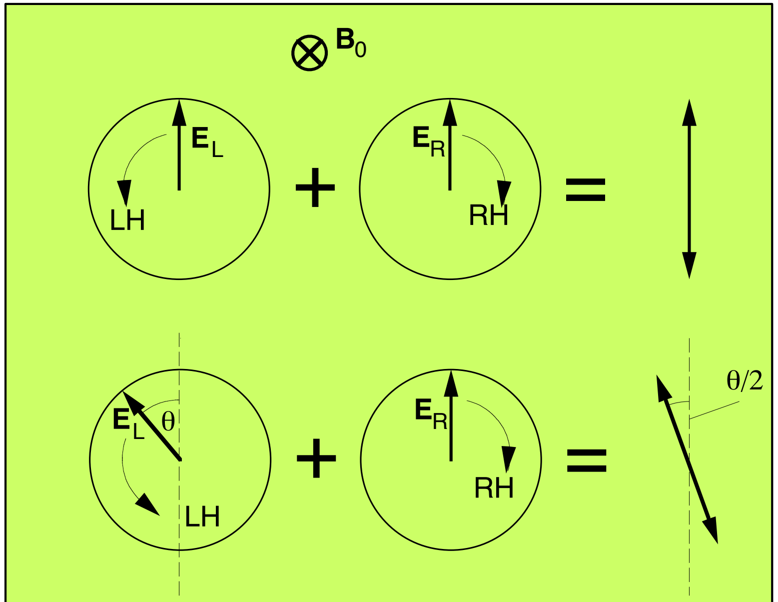
Waves in plasmas

High Frequency Electromagnetic waves propagating parallel to external B field

Faraday rotation

From the previous diagram it is clear that for large ω, the R wave travels faster than the L wave. Consider the plane polarised wave to be the sum of an R wave and an L wave (of course at the same frequency).

After let's say N cycles, the E_L and E_R vectors will return to their initial positions. However, after propagating a given distance d in a plasma the R and L waves will have undergone a different number of cycles since they require a different amount of time to cover the distance.



The principle of **Faraday rotation**

A plane polarised wave is the sum of left and right Handed circularly polarised waves

Since the L wave travels more slowly (in a plasma) it will have undergone N+a cycles at the position where R has undergone N cycles

The plane of polarisation is rotated

Waves in plasmas

High Frequency Electromagnetic waves propagating parallel to external B field

Faraday rotation

$$\theta(rad) = \frac{e^3 \lambda_o^2}{8\pi^2 m_e^2 \varepsilon_o c^3} \int_\ell \frac{n_e B_o}{\sqrt{1 - \frac{n_e}{n_c}}} dz \qquad \omega >> \omega_c$$

4.14
4.23
4.24
Chen

$$\frac{e^3}{8\pi^2 m_e^2 \varepsilon_o c^3} = 2.6312 \times 10^{-13} (\mathrm{T}^{-1})$$

$$\theta = \frac{e^3 \lambda_o^2}{8\pi^2 m_e^2 \varepsilon_o c^3} \int_\ell n_e B_o dz \qquad n_c >> n_e$$

$$n_c = \frac{\omega^2 m_e \varepsilon_o}{e^2} \qquad \omega_{pe}^2 = \frac{n_e e^2}{\varepsilon_o m_e} \qquad 1 - \frac{\omega_{pe}^2}{\omega^2} = 1 - \frac{n_e}{n_c}$$

Waves in plasmas

Electron waves (electrostatic):

$\boldsymbol{B}_o = 0$ or $\boldsymbol{k} // \boldsymbol{B}_o$

$$\omega^2 = \omega_p^2 + \frac{3}{2} k^2 \upsilon_{thermal}^2$$

Plasma oscillations

$\boldsymbol{k} \perp \boldsymbol{B}_o$

$$\omega^2 = \omega_p^2 + \omega_c^2$$

Upper hybrid oscillations

Ion waves (electrostatic):

$\boldsymbol{B}_o = 0$ or $\boldsymbol{k} // \boldsymbol{B}_o$

$$\omega^2 = k^2 \upsilon_s^2 = k^2 \left(\frac{\gamma_e k_B T_e + \gamma_\iota k_B T_\iota}{m_i} \right)^{1/2}$$

Acoustic waves

$\boldsymbol{k} \perp \boldsymbol{B}_o$

$$\omega^2 = \Omega_c^2 + k^2 \upsilon_s^2$$

Electrostatic ion cyclotron waves

or

$$\omega^2 = \omega_l^2 = \Omega_c \omega_c$$

Lower hybrid oscillations

Waves in plasmas

Electron waves (electromagnetic):

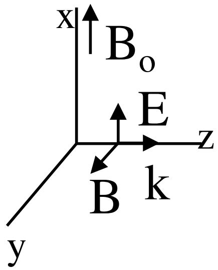
$\boldsymbol{B}_o = 0$	$\omega^2 = \omega_p^2 + k^2 c^2$	Light waves
$\boldsymbol{k} \perp \boldsymbol{B}_o, \boldsymbol{E}_1 // \boldsymbol{B}_o$	$\frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$	O - wave
$\boldsymbol{k} \perp \boldsymbol{B}_o, \boldsymbol{E}_1 \perp \boldsymbol{B}_o$	$\frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2}$	X - wave
$\boldsymbol{k} // \boldsymbol{B}_o$	$\frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2 - \omega\omega_c}$	R - wave whistler mode
	$\frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2 + \omega\omega_c}$	L - wave

Waves in plasmas

Ion waves (electromagnetic):		
$B_o = 0$	no	There is no electromagnetic wave
$k // B_o$	$\omega^2 = k^2 \upsilon_A^2$	Alfvèn wave
$k \perp B_o$	$\frac{\omega^2}{k^2} = c^2 \frac{\upsilon_s^2 + \upsilon_A^2}{c^2 + \upsilon_A^2}$	Magnetosonic wave

The above dispersion relations cover the main propagation geometries

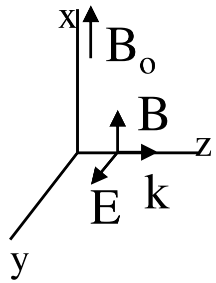
e.m. waves in a magnetised plasma



$\mathbf{k} \perp \mathbf{B_o}, \mathbf{E} \parallel \mathbf{B_o}$

$$\eta^2 = 1 - \frac{\omega_p^2}{\omega^2}$$

O-wave

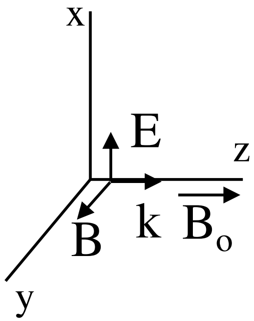


$\mathbf{k} \perp \mathbf{B_o}, \mathbf{E} \perp \mathbf{B_o}$

$$\eta^2 = 1 - \frac{\omega_p^2}{\omega^2} \frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_p^2 - \omega_c^2}$$

X-wave

Bire-fringence



$\mathbf{k} \parallel \mathbf{B_o}$

$$\eta^2 = 1 - \frac{\omega_p^2 / \omega^2}{1 \mp \omega_c / \omega}$$

(-) R-wave

(+) L-wave

Faraday rotation

$$\omega_c = \frac{eB_o}{m_e} \qquad \omega_{pe} = \left(\frac{n_e e^2}{\varepsilon_o m_e} \right)^{1/2}$$

Erasmus+