Co-funded by the Erasmus+ Programme of the European Union

Linear System of Equations

An Introduction – A World in Notation

Systems of linear equations are common in science and mathematics. A **system of linear equations** (or **linear system**) is a collection of one or more linear equations involving the same set of variables.

A Linear equation

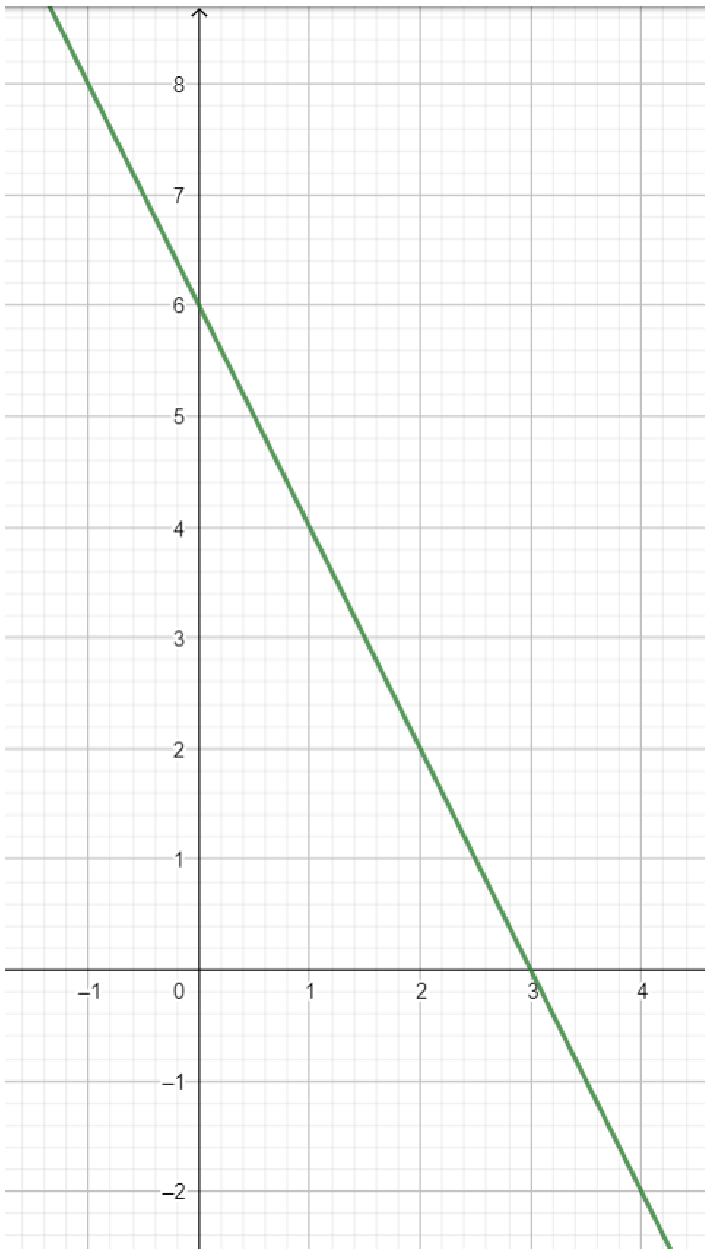
- There are problems where their solution lead us in an equation with two unknowns x, y, of the following form

$$a x + b y = c$$

- Solution of the equation $a x + b y = c$ is called any pair of numbers (x, y) that verifies it.

1st Example

- The equation $2x + y = 6$
- We note that for $x=1$ and $y=4$ the equation is verified, but for $x=2$ and $y=1$ is not verified.
- The pair of numbers (1, 4) that verifies the equation $2x + y = 6$ we say that is one of its solution.
- The equation $2x + y = 6$ does not have only the solution (1, 4), but it has infinite solutions.



2nd Example

- Suppose we have two linear equations with two unknowns

$$x + y = 5$$

$$2x + y = 8$$

- Solution of a linear system of two equations with two unknowns x and y is called each pair (x, y) that verifies its equations.

Graphic solution of a linear system with two unknowns

- For the graphical solution of a 2x2 linear system we work as follows

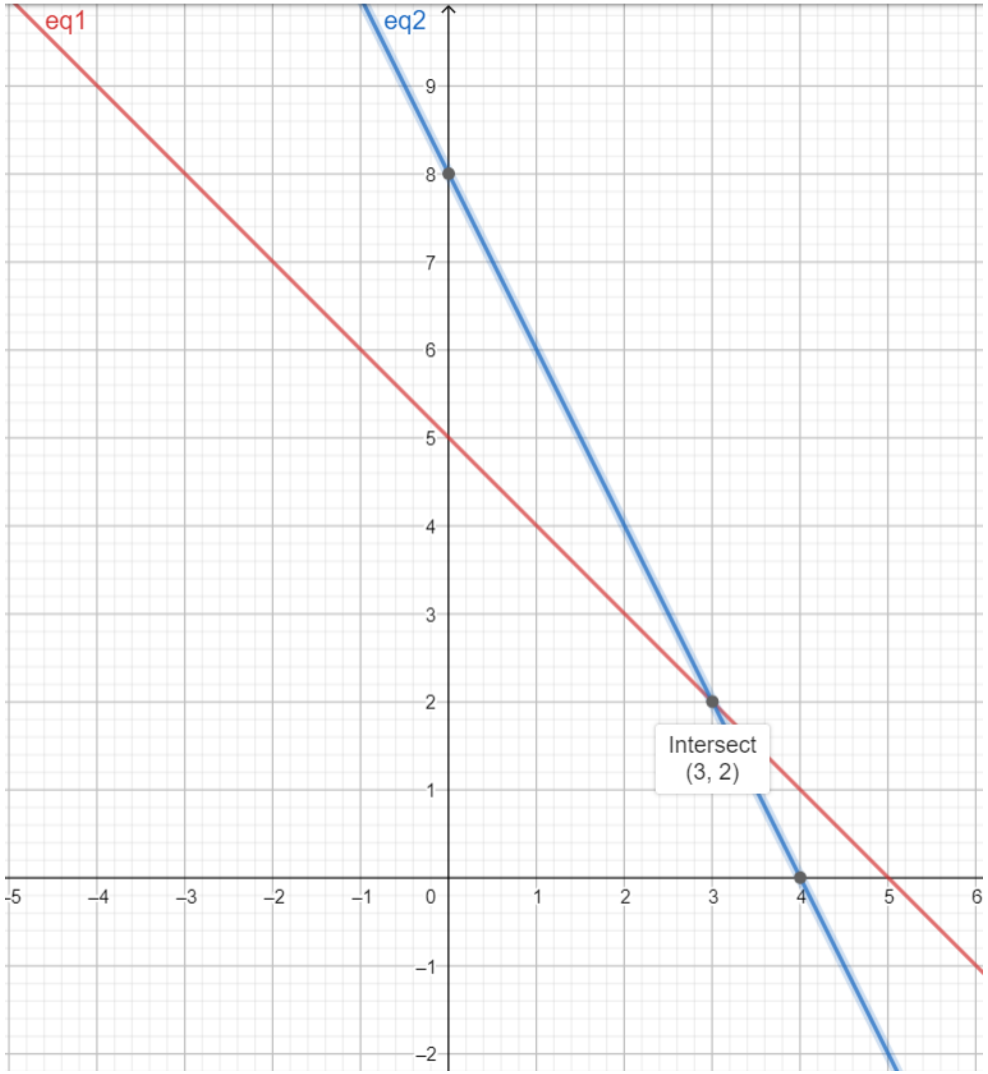
$$x + y = 5$$

$$2x + y = 8$$

- We draw in **Geogebra** the lines in the same axis system

$$\text{Eq(1)}: \quad x + y = 5$$

$$\text{Eq(2)}: \quad 2x + y = 8$$



Co-funded by the Erasmus+ Programme of the European Union

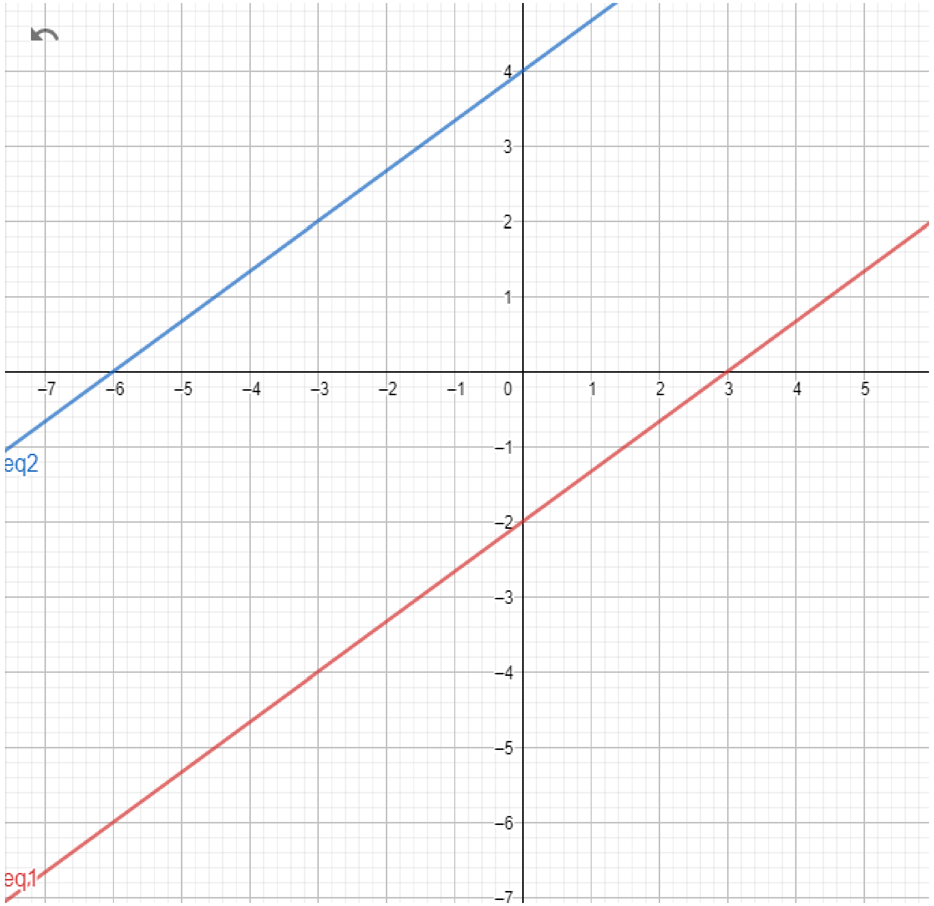
- We observe that **the lines intersect at the point (3, 2).** This point belongs to both lines and the coordinates of x = 3 and y = 2 verify both equations of the system. So the pair (3, 2) is a solution of the system.
- **These lines have no other common point**, so the system has no other solution. This means that the pair (3, 2) is the **only solution** in the system.

3rd Example

- Suppose now that we have the following system:

$$2x - 3y = 6$$

$$4x - 6y = -24$$



- To solve the system we draw the lines. We notice in the next figure that **the lines are parallel**.
- This means that **they have no common point**, so **the system has no solution**. In this case we say that **the system is impossible**.

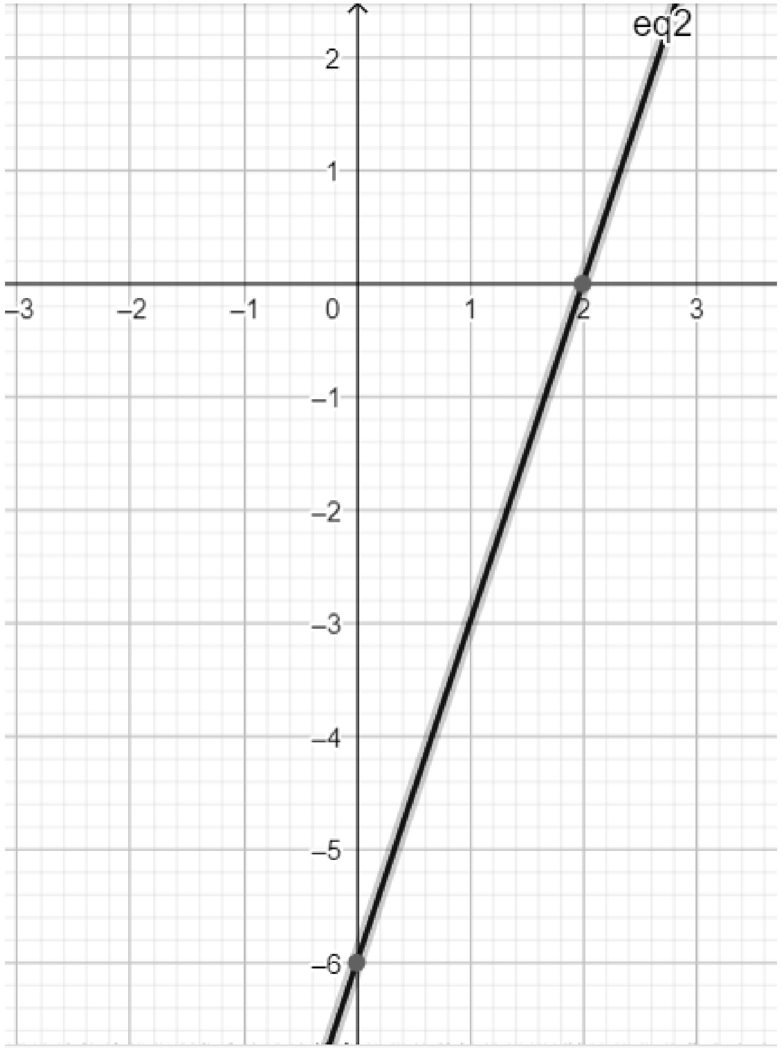
4th Example

- Suppose now that we have the following system:

$$3x - y = 6$$

$$6x - 2y = 12$$

- To solve the system we draw the lines. We notice in the figure on the right, that the lines are the same.
- So **they have all their points in common** and therefore **the system has infinite solutions**.



Algebraic solution of a linear system

Elimination of variables (2x2 system)

The simplest method for solving a system of linear equations is to repeatedly eliminate variables. This method can be described as follows:

- In the first equation, solve for one of the variables in terms of the others.
- Substitute this expression into the other equation. Thus, a single linear equation.
- Solve this equation, and then back-substitute into the previous equation.
- The solution is found.

5^{th} Example

- Solve the following system of linear equations

$$x + y = 20$$

$$x + 3y = 44$$

- Solve the equation $x + y = 20$ with respect to x,
- So we have $x = 20 - y$
- We substitute x with $20 - y$ into the other equation $x + 3y = 44$

$(20 – y) + 3y = 44$

$20 + 2y = 44$

$2y = 44 - 20$

$2y = 24$ thus $\mathbf{y = 12}$

For $y = 12$ and from $x = 20 – y$ we obtain

$x = 20 - 12$

$\mathbf{x = 8}$

Co-funded by the Erasmus+ Programme of the European Union

Verification

- So the solution is x = 8 and y = 12

- For verification we substitute the values of x and y into original system of equations

$$x + y = 20$$

$$x + 3y = 44$$

We get

$$8 + 12 = 20 \text{ and}$$

$$8 + 3 * 12 = 8 + 36 = 44$$

Algebraic solution of a linear system

Elimination of variables (Generalization in nxn system)

The simplest method for solving a system of linear equations is to repeatedly eliminate variables. This method can be described as follows:

- In the first equation, solve for one of the variables in terms of the others.
- Substitute this expression into the remaining equations. This yields a system of equations with one fewer equation and one fewer unknown.
- Repeat until the system is reduced to a single linear equation.
- Solve this equation, and then back-substitute until the entire solution is found.

6th Example

The system

$$\begin{aligned} 3x_1 + 2x_2 + \ \ x_3 &= 1 \\ x_2 - \ \ x_3 &= 2 \\ 2x_3 &= 4 \end{aligned}$$

is in strict triangular form, since in the second equation the coefficients are 0, 1, −1, respectively, and in the third equation the coefficients are 0, 0, 2, respectively. Because of the strict triangular form, the system is easy to solve. It follows from the third equation that $x_3 = 2$. Using this value in the second equation, we obtain

$$x_2 - 2 = 2 \qquad \text{or} \qquad x_2 = 4$$

Using $x_2 = 4$, $x_3 = 2$ in the first equation, we end up with

$$\begin{aligned} 3x_1 + 2 \cdot 4 + \ 2 &= \ \ 1 \\ x_1 &= -3 \end{aligned}$$

Thus, the solution of the system is $(-3, 4, 2)$.

7th Example from Electrical Circuits

Solve for the current flowing through the 8Ω resistor

Therefore, we will solve for $i_{8\Omega}$ using nodal analysis.

4. ***Attempt** a problem solution*. We first write down all of the equations we will need in order to find $i_{8\Omega}$.

$$i_{8\Omega} = i_2, \qquad i_2 = \frac{v_1}{8}, \qquad i_{8\Omega} = \frac{v_1}{8}$$

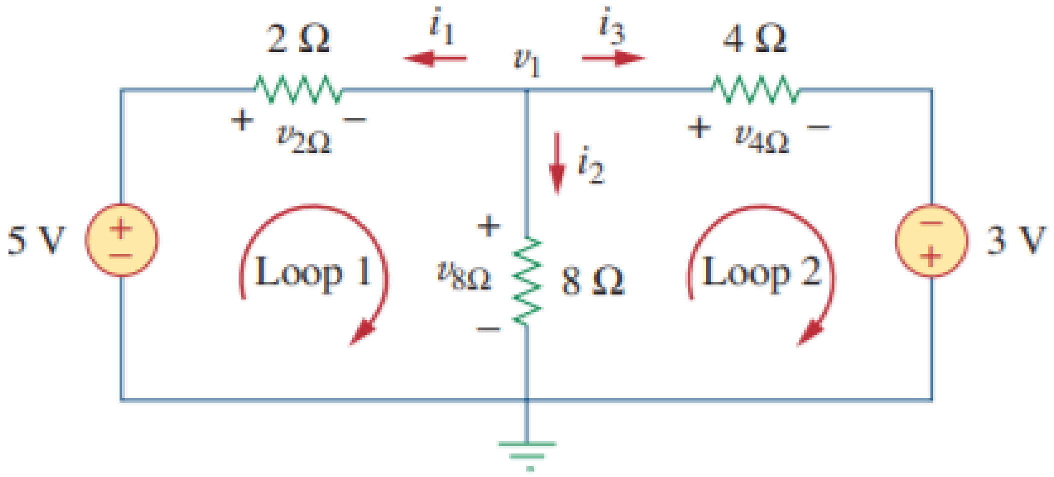
$$\frac{v_1 - 5}{2} + \frac{v_1 - 0}{8} + \frac{v_1 + 3}{4} = 0$$

Now we can solve for v_1.

$$8\left[\frac{v_1 - 5}{2} + \frac{v_1 - 0}{8} + \frac{v_1 + 3}{4}\right] = 0$$

leads to $(4v_1 - 20) + (v_1) + (2v_1 + 6) = 0$

$$7v_1 = +14, \qquad v_1 = +2\text{ V}, \qquad i_{8\Omega} = \frac{v_1}{8} = \frac{2}{8} = \mathbf{0.25\ A}$$



5. ***Evaluate** the solution and check for accuracy.* We can now use Kirchhoff's voltage law (KVL) to check the results.

$$i_1 = \frac{v_1 - 5}{2} = \frac{2 - 5}{2} = -\frac{3}{2} = -1.5 \text{ A}$$

$$i_2 = i_{8\Omega} = 0.25 \text{ A}$$

$$i_3 = \frac{v_1 + 3}{4} = \frac{2 + 3}{4} = \frac{5}{4} = 1.25 \text{ A}$$

$$i_1 + i_2 + i_3 = \mathbf{-1.5 + 0.25 + 1.25 = 0} \qquad \text{(Checks.)}$$

8^{th} Example from Electrical Circuits

Assume an electric network consisting of two voltage sources and three resistors.

According to the first law:

$$i_1 - i_2 - i_3 = 0$$

Applying the second law to the closed circuit s_1, and substituting for voltage using Ohm's law gives:

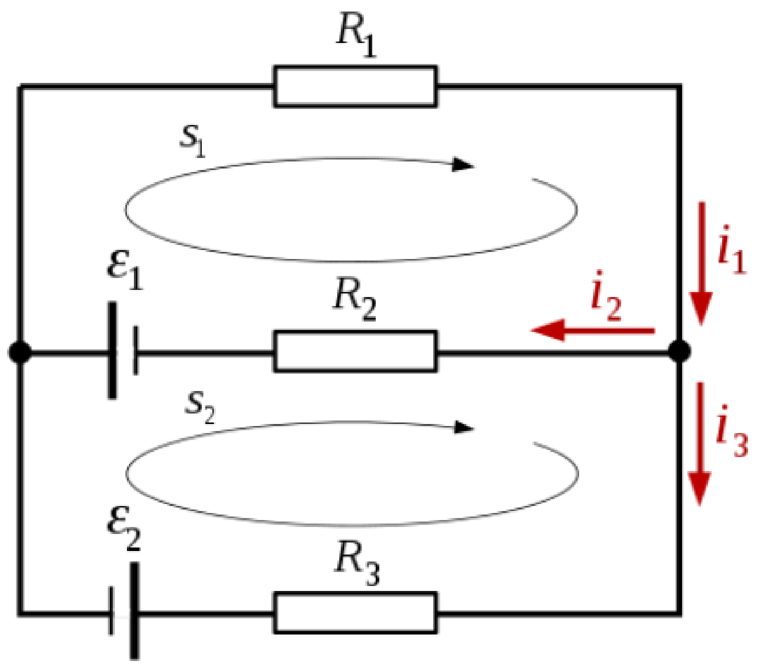
$$-R_2 i_2 + \mathcal{E}_1 - R_1 i_1 = 0$$

The second law, again combined with Ohm's law, applied to the closed circuit s_2 gives:

$$-R_3 i_3 - \mathcal{E}_2 - \mathcal{E}_1 + R_2 i_2 = 0$$

This yields a system of linear equations in i_1, i_2, i_3:

$$\begin{cases} i_1 - i_2 - i_3 & = 0 \\ -R_2 i_2 + \mathcal{E}_1 - R_1 i_1 & = 0 \\ -R_3 i_3 - \mathcal{E}_2 - \mathcal{E}_1 + R_2 i_2 & = 0 \end{cases}$$



which is equivalent to

$$\begin{cases} i_1 + (-i_2) + (-i_3) & = 0 \\ R_1 i_1 + R_2 i_2 + 0 i_3 & = \mathcal{E}_1 \\ 0 i_1 + R_2 i_2 - R_3 i_3 & = \mathcal{E}_1 + \mathcal{E}_2 \end{cases}$$

Assuming

$$R_1 = 100\Omega,\ R_2 = 200\Omega,\ R_3 = 300\Omega$$

$$\mathcal{E}_1 = 3\mathrm{V}, \mathcal{E}_2 = 4\mathrm{V}$$

- Thus the system of linear equation becomes

$$
\begin{aligned}
i_1 - \quad i_2 - \quad i_3 &= 0 \\
100 \cdot i_1 + 200 \cdot i_2 + \quad 0 \cdot i_3 &= 3 \\
0 \cdot i_1 + 200 \cdot i_2 - 300 \cdot i_3 &= 7
\end{aligned}
$$

$$\Delta = \begin{vmatrix} 1 & -1 & -1 \\ 100 & 200 & 0 \\ 0 & 200 & -300 \end{vmatrix} = -110000$$

$$\Delta_1 = \begin{vmatrix} 0 & -1 & -1 \\ 3 & 200 & 0 \\ 7 & 200 & -300 \end{vmatrix} = -100;$$

$$\Delta_2 = \begin{vmatrix} 1 & 0 & -1 \\ 100 & 3 & 0 \\ 0 & 7 & -300 \end{vmatrix} = -1600$$

Co-funded by the Erasmus+ Programme of the European Union

$$\Delta_3 = \begin{vmatrix} 1 & -1 & 0 \\ 100 & 200 & 3 \\ 0 & 200 & 7 \end{vmatrix} = 1500$$

$$x_1 = \Delta_1 / \Delta = \frac{-100}{-110000} = \frac{1}{1100}$$

$$x_2 = \Delta_2 / \Delta = \frac{-1600}{-110000} = \frac{4}{275}$$

$$x_3 = \Delta_3 / \Delta = \frac{1500}{-110000} = \frac{-3}{220}$$

the solution is

$$\begin{cases} i_1 = \frac{1}{1100}\,\mathrm{A} \\ i_2 = \frac{4}{275}\,\mathrm{A} \\ i_3 = -\frac{3}{220}\,\mathrm{A} \end{cases}$$

The current i_3 has a negative sign which means the assumed direction of i_3 was incorrect and i_3 is actually flowing in the direction opposite to the red arrow labeled i_3. The current in R_3 flows from left to right.

9th Example

Solve the system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 - x_2 - 3x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 4 \end{aligned}$$

Solution

Subtracting 3 times the first row from the second row yields

$$-7x_2 - 6x_3 = -10$$

Subtracting 2 times the first row from the third row yields

$$-x_2 - x_3 = -2$$

If the second and third equations of our system, respectively, are replaced by these new equations, we obtain the equivalent system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ -7x_2 - 6x_3 &= -10 \\ -x_2 - x_3 &= -2 \end{aligned}$$

If the third equation of this system is replaced by the sum of the third equation and $-\frac{1}{7}$ times the second equation, we end up with the following strictly triangular system:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ -7x_2 - 6x_3 &= -10 \\ -\tfrac{1}{7}x_3 &= -\tfrac{4}{7} \end{aligned}$$

Using back substitution, we get

$$x_3 = 4, \qquad x_2 = -2, \qquad x_1 = 3$$

Matrix

- A *matrix* is a rectangular array of numbers or other mathematical objects for which operations such as addition and multiplication are defined.
- Most commonly, a matrix over a field *F* is a rectangular array of scalars each of which is a member of *F*.
- Here we focuses on *real* and *complex matrices*, that is, matrices whose elements are real numbers or complex numbers, respectively. For instance, this is a real matrix:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix}$$

- The numbers, symbols, or expressions in the matrix are called its entries or its elements. The horizontal and vertical lines of entries in a matrix are called rows and columns, respectively.

Co-funded by the Erasmus+ Programme of the European Union

Innovative Teaching Education in Mathematics

Types of Matrices

Row vector	1 × *n*	$\begin{bmatrix} 3 & 7 & 2 \end{bmatrix}$	A matrix with one row, sometimes used to represent a vector
Column vector	*n* × 1	$\begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$	A matrix with one column, sometimes used to represent a vector
Square matrix	*n* × *n*	$\begin{bmatrix} 9 & 13 & 5 \\ 1 & 11 & 7 \\ 2 & 6 & 3 \end{bmatrix}$	

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

An $m \times n$ matrix: the m rows are horizontal and the n columns are vertical. Each element of a matrix is often denoted by a variable with two subscripts. For example, $a_{2,1}$ represents the element at the second row and first column of the matrix.

Addition and Scalar multiplication

Addition	The *sum* **A**+**B** of two *m*-by-*n* matrices **A** and **B** is calculated entrywise: $(\mathbf{A}+\mathbf{B})_{i,j} = \mathbf{A}_{i,j} + \mathbf{B}_{i,j}$, where $1 \le i \le m$ and $1 \le j \le n$.

$$\begin{bmatrix}1 & 3 & 1\\ 1 & 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 5\\ 7 & 5 & 0\end{bmatrix} = \begin{bmatrix}1+0 & 3+0 & 1+5\\ 1+7 & 0+5 & 0+0\end{bmatrix} = \begin{bmatrix}1 & 3 & 6\\ 8 & 5 & 0\end{bmatrix}$$

The product *c***A** of a number *c* (also called a scalar in the parlance of abstract algebra) and a matrix **A** is computed by multiplying every entry of **A** by *c*:

$$(c\mathbf{A})_{i,j} = c \cdot \mathbf{A}_{i,j}.$$

$$2 \cdot \begin{bmatrix}1 & 8 & -3\\ 4 & -2 & 5\end{bmatrix} = \begin{bmatrix}2\cdot 1 & 2\cdot 8 & 2\cdot -3\\ 2\cdot 4 & 2\cdot -2 & 2\cdot 5\end{bmatrix} = \begin{bmatrix}2 & 16 & -6\\ 8 & -4 & 10\end{bmatrix}$$

Co-funded by the Erasmus+ Programme of the European Union

Matrix Transposition

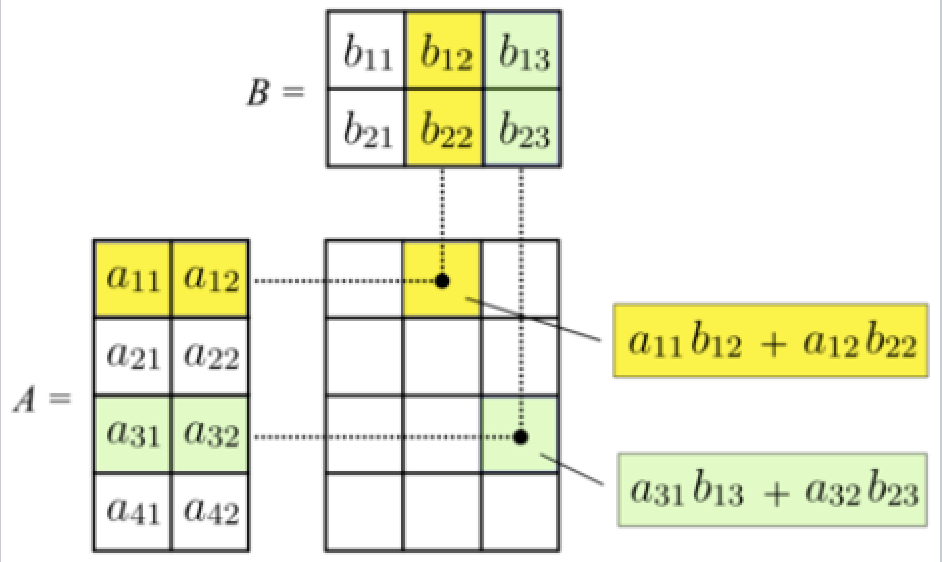
Transposition	The *transpose* of an *m*-by-*n* matrix **A** is the *n*-by-*m* matrix $\mathbf{A}^T$ (also denoted $\mathbf{A}^{tr}$ or ${}^t\mathbf{A}$) formed by turning rows into columns and vice versa: $(\mathbf{A}^T)_{i,j} = \mathbf{A}_{j,i}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & 7 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & -6 \\ 3 & 7 \end{bmatrix}$$

Familiar properties of numbers extend to these operations of matrices: for example, addition is commutative, that is, the matrix sum does not depend on the order of the summands: A + B = B + A.
The transpose is compatible with addition and scalar multiplication, as expressed by $(cA)^T = c(A^T)$ and $(A + B)^T = A^T + B^T$. Finally, $(A^T)^T = A$.

Matrix multiplication

- *Multiplication* of two matrices is defined if and only if the number of columns of the left matrix is the same as the number of rows of the right matrix.
- If **A** is an *m*-by-*n* matrix and **B** is an *n*-by-*p* matrix, then their *matrix product* **AB** is the *m*-by-*p* matrix whose entries are given by dot product of the corresponding row of **A** and the corresponding column of **B**:



Schematic depiction of the matrix product **AB** of two matrices **A** and **B**.

$$[\mathbf{AB}]_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \cdots + a_{i,n}b_{n,j} = \sum_{r=1}^{n} a_{i,r}b_{r,j},$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1000 \\ 1 & 100 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 3 & 2340 \\ 0 & 1000 \end{bmatrix}$$

- Matrix multiplication satisfies the rules (**AB**)**C** = **A**(**BC**) (associativity), and (**A** + **B**)**C** = **AC** + **BC** as well as **C**(**A** + **B**) = **CA** + **CB** (left and right distributivity), whenever the size of the matrices is such that the various products are defined.
- The product **AB** may be defined without **BA** being defined, namely if **A** and **B** are m-by-n and n-by-k matrices, respectively, and m ≠ k. Even if both products are defined, they need not be equal, that is, generally

- **AB ≠ BA**,
- that is, matrix multiplication is not commutative, in marked contrast to (rational, real, or complex) numbers whose product is independent of the order of the factors. An example of two matrices not commuting with each other is:

- An example of two matrices not commuting with each other is:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix},$$

whereas

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}.$$

Submatrix

- A **submatrix** of a matrix is obtained by deleting any collection of rows and/or columns.
- For example, from the following 3-by-4 matrix, we can construct a 2-by-3 submatrix by removing row 3 and column 2:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 5 & 7 & 8 \end{bmatrix}.$$

Square matrix

- A square matrix is a matrix with the same number of rows and columns. An *n*-by-*n* matrix is known as a square matrix of order *n.* Any two square matrices of the same order can be added and multiplied. The entries a_{ii} form the main diagonal of a square matrix. They lie on the imaginary line that runs from the top left corner to the bottom right corner of the matrix.

Diagonal and triangular matrix

- If all entries of **A** below the main diagonal are zero, **A** is called an *upper triangular matrix*.
- Similarly if all entries of *A* above the main diagonal are zero, **A** is called a *lower triangular matrix*.
- If all entries outside the main diagonal are zero, **A** is called a diagonal matrix.

Name	Example with *n* = 3
Diagonal matrix	$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$
Lower triangular matrix	$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
Upper triangular matrix	$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

Identity matrix

- The *identity matrix* $\mathbf{I}_n$ of size *n* is the *n*-by-*n* matrix in which all the elements on the main diagonal are equal to 1 and all other elements are equal to 0, for example,

$$\mathbf{I}_1 = [1],\ \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \cdots,\ \mathbf{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Invertible matrix and its inverse

- A square matrix **A** is called invertible or non-singular if there exists a matrix **B** such that

- $\mathbf{AB} = \mathbf{BA} = \mathbf{I_n}$
- where $\mathbf{I_n}$ is the n×n identity matrix with 1s on the main diagonal and 0s elsewhere. If **B** exists, it is unique and is called the inverse matrix of **A**, denoted $\mathbf{A^{-1}}$.

Determinant

- The *determinant* det(**A**) or |**A**| of a square matrix **A** is a number encoding certain properties of the matrix.
- A matrix is invertible if and only if its determinant is nonzero.

Determinant

In the case of a 2 × 2 matrix the determinant may be defined as

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Similarly, for a 3 × 3 matrix *A*, its determinant is

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} \square & \square & \square \\ \square & e & f \\ \square & h & i \end{vmatrix} - b\begin{vmatrix} \square & \square & \square \\ d & \square & f \\ g & \square & i \end{vmatrix} + c\begin{vmatrix} \square & \square & \square \\ d & e & \square \\ g & h & \square \end{vmatrix}$$

$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= aei + bfg + cdh - ceg - bdi - afh.$$

General form

A general system of *m* linear equations with *n* unknowns can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

where $x_1, x_2, \ldots, x_n$ are the unknowns, $a_{11}, a_{12}, \ldots, a_{mn}$ are the coefficients of the system, and $b_1, b_2, \ldots, b_m$ are the constant terms.

Matrix equation

The vector equation is equivalent to a matrix equation of the form

$$A\mathbf{x} = \mathbf{b}$$

where A is an $m \times n$ matrix, $\mathbf{x}$ is a column vector with n entries, and $\mathbf{b}$ is a column vector with m entries.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

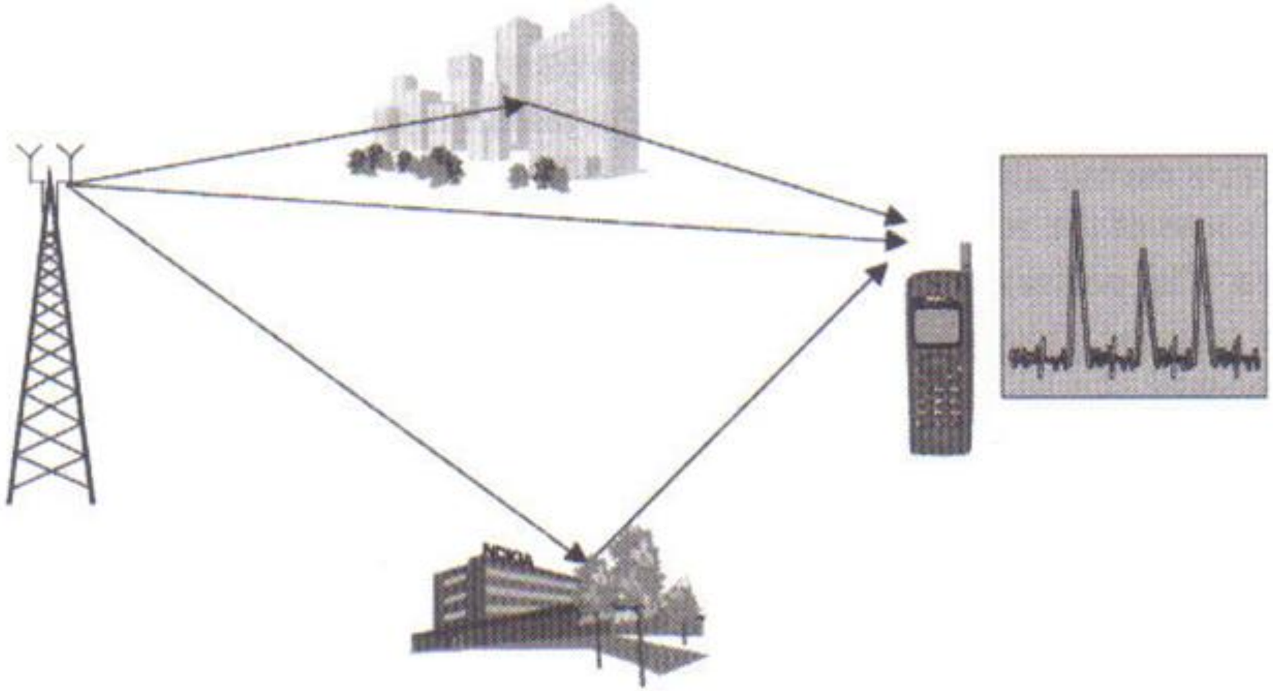
Cramer's rule

$$\begin{aligned} x + 3y - 2z &= 5 \\ 3x + 5y + 6z &= 7 \\ 2x + 4y + 3z &= 8 \end{aligned}$$

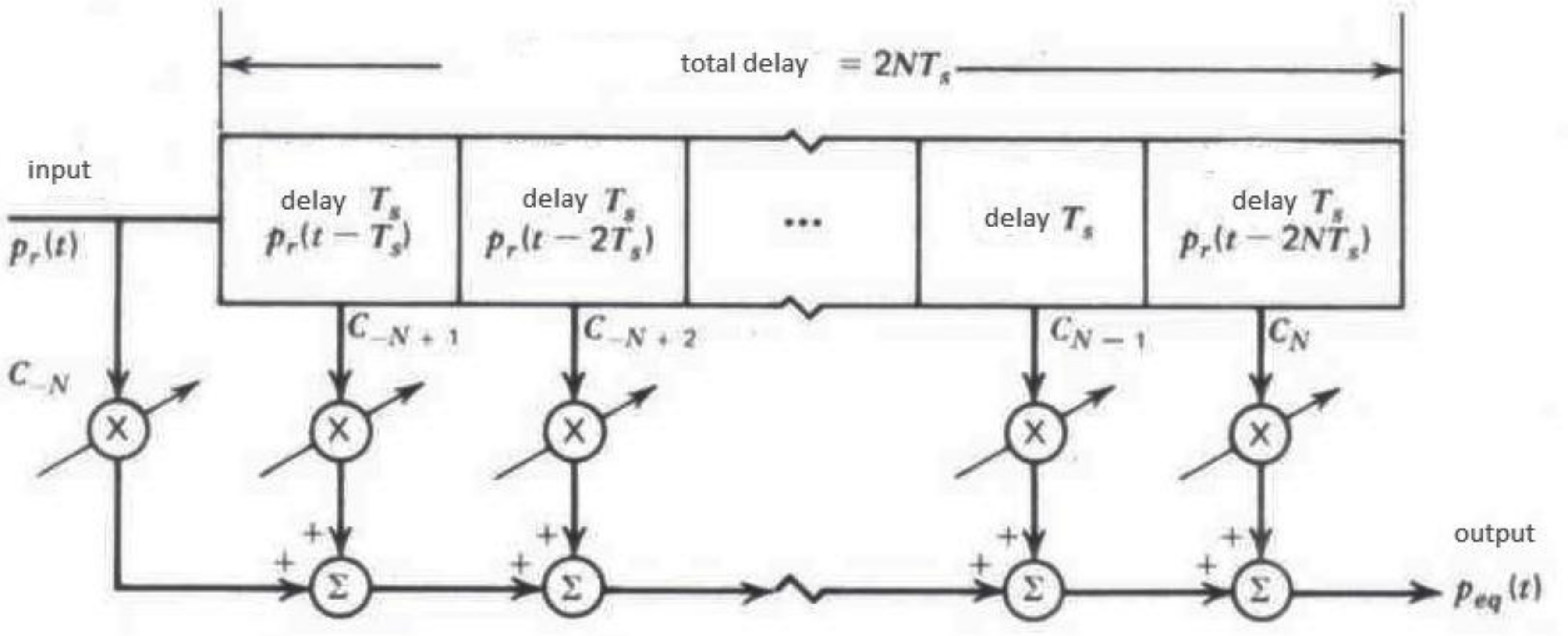
is given by

$$x = \frac{\begin{vmatrix} 5 & 3 & -2 \\ 7 & 5 & 6 \\ 8 & 4 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} 1 & 5 & -2 \\ 3 & 7 & 6 \\ 2 & 8 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 2 & 4 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}}.$$

Multipath propagation leads to ISI



Zero-forcing equalizer



- We will assume that the filter has (2N +1) taps with gains C_{-N}, C_{-N+1}, ..., C_0, C_1,. . . C_N. The input of the equalizer is $p_r(t)$ that is known, and the output is $p_{eq}(t)$. We can write the output $p_{eq}(t)$ as a function of $p_r(t)$ and the gains of the taps as

$$p_{eq}(k) = \sum_{n=-N}^{N} C_n p_r(k-n)$$

$$p_{eq}(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k = \pm 1, \pm 2, \ldots, \pm N \end{cases}$$

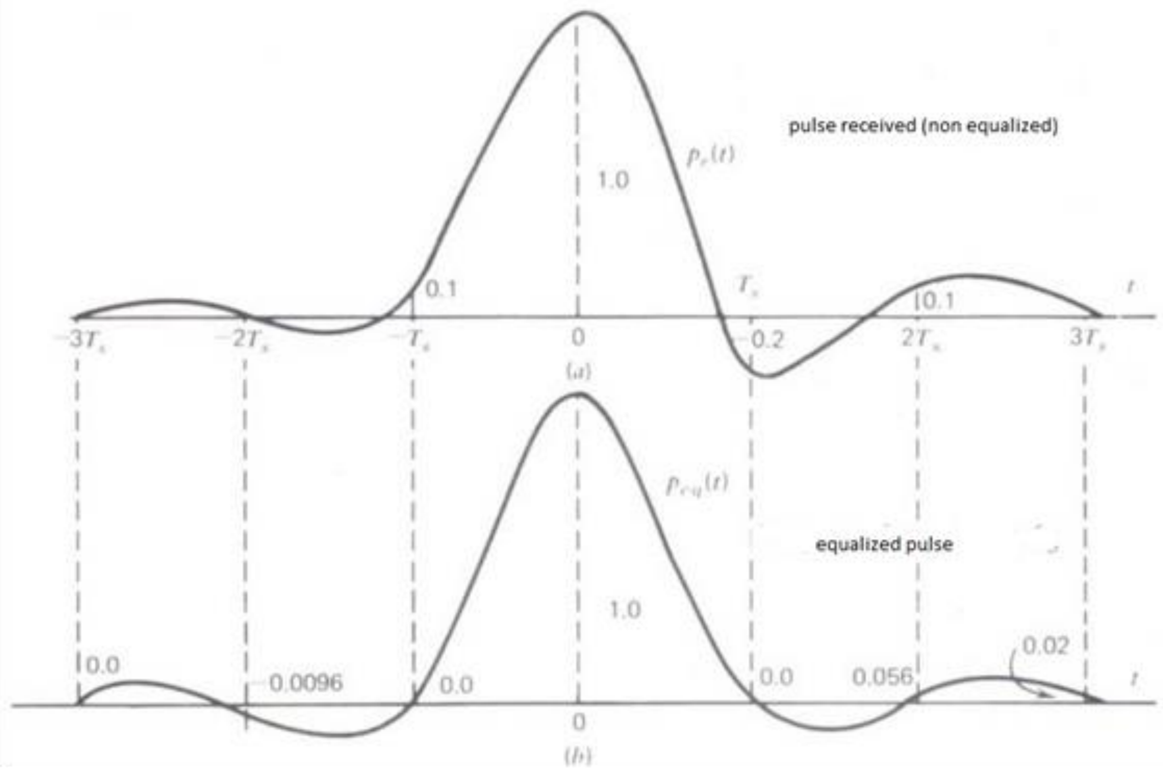
- By combining the equations above, we have

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} p_r(0) & p_r(-1) & \cdots & p_r(-2N) \\ p_r(1) & p_r(0) & \cdots & p_r(-2N+1) \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ p_r(2N) & \cdots & \cdots & p_r(0) \end{bmatrix} \begin{bmatrix} C_{-N} \\ C_{-N+1} \\ \vdots \\ \vdots \\ C_0 \\ \\ C_{N-1} \\ C_N \end{bmatrix}$$

This equation represents a system of (2N + 1) equations and can be solved with respect to C_n. The equalizer describing the system is called a zero forcing equalizer because $p_{eq}(k)$ has N zero values on each side.

Example: How the 3 tap equalizer works

- Design a three-tap equalizer to reduce the ISI due to the non equalized pulse $p_r(t)$ of the figure below



Solution

- At $t = 0$, the current pulse is sampled, after T_s sec the next sampling pulse will be taken, etc. Thus, we want the current pulse to be zero in the integer multiples of T_s so that there are no residual current pulses at the next points sampling. That's what we want to apply to every pulse.
- Observing the non-equalized pulse and the equalized below, we see that this equalizer forces the pulse to zero at the sampling points of the next T_s and the previous pulse $-T_s$, because it is three tap equalizer. If he was five tap equalizer, it would force the pulse to become zero at the T_s, $2T_s$ and $-T_s$, $-2T_s$ points.

- With a three tap equalizer here, we can cause a "zeroing" in the equalized pulse both before and after the point t = 0. The gains of the taps for this equalizer are given by the solution of the following 3x3 system of equations which is given below with the form of a matrix equation

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.0 & 0.1 & 0 \\ -0.2 & 1.0 & 0.1 \\ 0.1 & -0.2 & 1 \end{bmatrix} \begin{bmatrix} C_{-1} \\ C_0 \\ C_1 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 1 & \frac{1}{10} & 0 \\ \frac{-1}{5} & 1 & \frac{1}{10} \\ \frac{1}{10} & \frac{-1}{5} & 1 \end{vmatrix} = \frac{1041}{1000}$$

$$\Delta_2 = \begin{vmatrix} 1 & 0 & 0 \\ \frac{-1}{5} & 1 & \frac{1}{10} \\ \frac{1}{10} & 0 & 1 \end{vmatrix} = 1$$

$$x_1 = \Delta_1 / \Delta = \frac{\frac{-1}{10}}{\frac{1041}{1000}} = \frac{-100}{1041}$$

$$x_2 = \Delta_2 / \Delta = \frac{1}{\frac{1041}{1000}} = \frac{1000}{1041}$$

$$x_3 = \Delta_3 / \Delta = \frac{\frac{21}{100}}{\frac{1041}{1000}} = \frac{70}{347}$$

$$\Delta_1 = \begin{vmatrix} 0 & \frac{1}{10} & 0 \\ 1 & 1 & \frac{1}{10} \\ 0 & \frac{-1}{5} & 1 \end{vmatrix} = \frac{-1}{10};$$

$$\Delta_3 = \begin{vmatrix} 1 & \frac{1}{10} & 0 \\ \frac{-1}{5} & 1 & 1 \\ \frac{1}{10} & \frac{-1}{5} & 0 \end{vmatrix} = \frac{21}{100};$$

$$\begin{bmatrix} C_{-1} \\ C_0 \\ C_1 \end{bmatrix} = \begin{bmatrix} -0.09606 \\ 0.9606 \\ 0.2017 \end{bmatrix}$$

- So, for these specific values of C_{-1}, C_0 and C_1 the three tap equalizer; equalize the pulse in order to reduce ISI.

References

- GeoGebra https://www.geogebra.org/calculator
- Solving systems of linear equations, https://matrixcalc.org/en/slu.html
- Matrix https://en.wikipedia.org/wiki/Matrix_(mathematics)
- System of linear equations https://en.wikipedia.org/wiki/System_of_linear_equations
- Kirchhoff's circuit laws

https://en.wikipedia.org/wiki/Kirchhoff%27s_circuit_laws

- Fundamentals of Electric Circuits

https://sabotin.ung.si/~mv0029/pdf/Alexander_2.pdf

Thank you!

Any Questions?