Chapter 1

FUNDAMENTALS OF VIBRATION

1.1 INTRODUCTION

Before beginning a discussion of acoustics, we should settle on a system of units. Acoustics encompasses such a wide range of scientific and engineering disciplines that the choice is not easy. A survey of the literature reveals a great lack of uniformity: writers use units common to their particular fields of interest. Most early work has been reported in the CGS (centimeter–gram–second) system, but considerable engineering work has been reported in a mixture of metric and English units. Work in electroacoustics and underwater acoustics has commonly been reported in the MKS (meter–kilogram–second) system. A codification of the MKS system, the SI (Le Système International d'Unités), has been established as the standard. This is the system generally used in this book. CGS and SI units are equated and compared in Appendix A1.

Throughout this text, "log" will represent logarithm to the base 10 and "ln" (the "natural logarithm") will represent logarithm to the base e.

Acoustics as a science may be defined as the generation, transmission, and reception of energy as vibrational waves in matter. When the molecules of a fluid or solid are displaced from their normal configurations, an internal elastic restoring force arises. It is this elastic restoring force, coupled with the inertia of the system, that enables matter to participate in oscillatory vibrations and thereby generate and transmit acoustic waves. Examples include the tensile force produced when a spring is stretched, the increase in pressure produced when a fluid is compressed, and the restoring force produced when a point on a stretched wire is displaced transverse to its length.

The most familiar acoustic phenomenon is that associated with the sensation of sound. For the average young person, a vibrational disturbance is interpreted as sound if its frequency lies in the interval from about 20 Hz to 20,000 Hz (1 Hz = 1 hertz = 1 cycle per second). However, in a broader sense acoustics also includes the *ultrasonic* frequencies above 20,000 Hz and the *infrasonic* frequencies below 20 Hz. The natures of the vibrations associated with acoustics are many, including the simple sinusoidal vibrations produced by a tuning fork, the complex vibrations generated by a bowed violin string, and the nonperiodic motions associated with an explosion, to mention but a few. In studying vibrations it is advisable to begin with the simplest type, a one-dimensional sinusoidal vibration that has only a single frequency component (a pure tone).

1.2 THE SIMPLE OSCILLATOR

If a mass m, fastened to a spring and constrained to move parallel to the spring, is displaced slightly from its rest position and released, the mass will vibrate. Measurement shows that the displacement of the mass from its rest position is a sinusoidal function of time. Sinusoidal vibrations of this type are called *simple harmonic vibrations*. A large number of vibrators used in acoustics can be modeled as simple oscillators. Loaded tuning forks and loudspeaker diaphragms, constructed so that at low frequencies their masses move as units, are but two examples. Even more complex vibrating systems have many of the characteristics of the simple systems and may often be modeled, to a first approximation, by simple oscillators.

The only physical restrictions placed on the equations for the motion of a simple oscillator are that the restoring force be directly proportional to the displacement (Hooke's law), the mass be constant, and there be no losses to attenuate the motion. When these restrictions apply, the frequency of vibration is independent of amplitude and the motion is simple harmonic.

A similar restriction applies to more complex types of vibration, such as the transmission of an acoustic wave through a fluid. If the acoustic pressures are so large that they no longer are proportional to the displacements of the particles of fluid, it becomes necessary to replace the normal acoustic equations with more general equations that are much more complicated. With sounds of ordinary intensity this is not necessary, for even the noise generated by a large crowd at a football game rarely causes the amplitude of motion of the air molecules to exceed one-tenth of a millimeter, which is within the limit given above. The amplitude of the shock wave generated by a large explosion is, however, well above this limit, and hence the normal acoustic equations are not applicable.

Returning to the simple oscillator shown in Fig. 1.2.1, let us assume that the restoring force f in newtons (N) can be expressed by the equation

$$f = -sx \qquad (1.2.1)$$

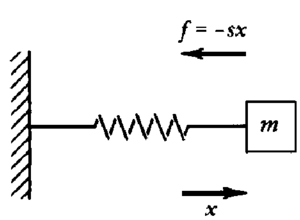


Figure 1.2.1 Schematic representation of a simple oscillator consisting of a mass m attached to one end of a spring of spring constant s. The other end of the spring is fixed.

where x is the displacement in meters (m) of the mass m in kilograms (kg) from its rest position, s is the *stiffness* or *spring constant* in N/m, and the minus sign indicates that the force is opposed to the displacement. Substituting this expression for force into the general equation of linear motion

$$f = m\frac{d^2x}{dt^2} \tag{1.2.2}$$

where d^2x/dt^2 is the acceleration of the mass, we obtain

$$\frac{d^2x}{dt^2} + \frac{s}{m}x = 0 \tag{1.2.3}$$

Both s and m are positive, so that we can define a constant

$$\boxed{\omega_0^2 = s/m} \tag{1.2.4}$$

which casts our equation into the form

$$\boxed{\frac{d^2x}{dt^2} + \omega_0^2 x = 0} \tag{1.2.5}$$

This is an important linear differential equation whose general solution is well known and may be obtained by several methods.

One method is to assume a trial solution of the form

$$x = A_1 \cos \gamma t \tag{1.2.6}$$

Differentiation and substitution into (1.2.5) shows that this is a solution if $\gamma = \omega_0$. It may similarly be shown that

$$x = A_2 \sin \omega_0 t \tag{1.2.7}$$

is also a solution. The complete general solution is the sum of these two,

$$x = A_1 \cos \omega_0 t + A_2 \sin \omega_0 t \tag{1.2.8}$$

where A_1 and A_2 are arbitrary constants and the parameter ω_0 is the *natural angular frequency* in radians per second (rad/s). Since there are 2π radians in one cycle, the *natural frequency* f_0 in hertz (Hz) is related to the natural angular frequency by

$$f_0 = \omega_0/2\pi \tag{1.2.9}$$

Note that either decreasing the stiffness or increasing the mass lowers the frequency. The *period* T of one complete vibration is given by

$$T = 1/f_0 \tag{1.2.10}$$

1.3 INITIAL CONDITIONS

If at time $t = 0$ the mass has an initial displacement x_0 and an initial speed u_0, then the arbitrary constants A_1 and A_2 are fixed by these initial conditions and the subsequent motion of the mass is completely determined. Direct substitution into (1.2.8) of $x = x_0$ at $t = 0$ will show that A_1 equals the initial displacement x_0. Differentiation of (1.2.8) and substitution of the initial speed at $t = 0$ gives $u_0 = \omega_0 A_2$, and (1.2.8) becomes

$$x = x_0 \cos \omega_0 t + (u_0/\omega_0) \sin \omega_0 t \tag{1.3.1}$$

Another form of (1.2.8) may be obtained by letting $A_1 = A \cos \phi$ and $A_2 = -A \sin \phi$, where A and ϕ are two new arbitrary constants. Substitution and simplification then gives

$$x = A \cos(\omega_0 t + \phi) \tag{1.3.2}$$

where A is the *amplitude* of the motion and ϕ is the *initial phase angle* of the motion. The values of A and ϕ are determined by the initial conditions and are

$$A = [x_0^2 + (u_0/\omega_0)^2]^{1/2} \quad \text{and} \quad \phi = \tan^{-1}(-u_0/\omega_0 x_0) \tag{1.3.3}$$

Successive differentiation of (1.3.2) shows that the speed of the mass is

$$u = -U \sin(\omega_0 t + \phi) \tag{1.3.4}$$

where $U = \omega_0 A$ is the *speed amplitude*, and the acceleration of the mass is

$$a = -\omega_0 U \cos(\omega_0 t + \phi) \tag{1.3.5}$$

In these forms it is seen that the displacement lags 90° ($\pi/2$ rad) behind the speed and that the acceleration is 180° (π rad) out of phase with the displacement, as shown in Fig. 1.3.1.

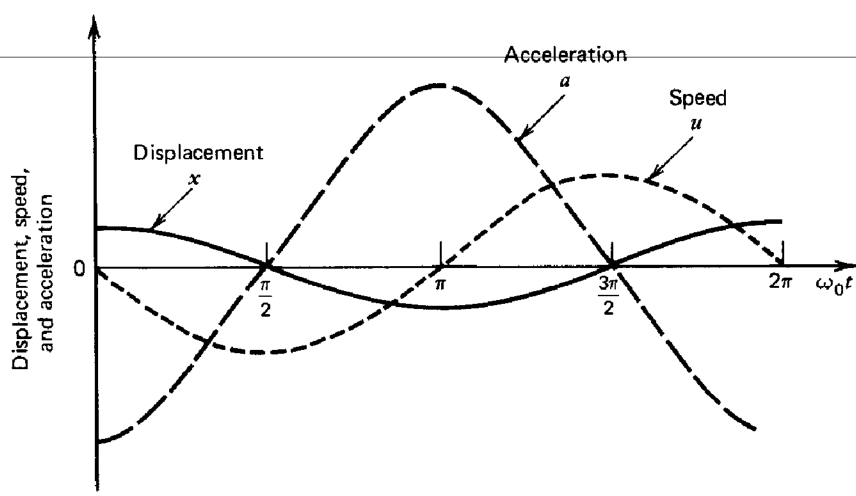


Figure 1.3.1 The speed u of a simple oscillator always leads to the displacement x by 90°. Acceleration a and displacement x are always 180° out of phase with each other. Plotted curves correspond to $\phi = 0°$.

1.4 ENERGY OF VIBRATION

The mechanical energy E of a system is the sum of the system's potential energy E_p and kinetic energy E_k. The potential energy is the work done in distorting the spring as the mass moves from its position of static equilibrium. Since the force exerted by the mass on the spring is in the direction of the displacement and equals $+sx$, the potential energy E_p stored in the spring is

$$E_p = \int_0^x sx\,dx = \tfrac{1}{2}sx^2 \qquad (1.4.1)$$

Expression of x by (1.3.2) gives

$$E_p = \tfrac{1}{2}sA^2\cos^2(\omega_0 t + \phi) \qquad (1.4.2)$$

The kinetic energy possessed by the mass is

$$E_k = \tfrac{1}{2}mu^2 \qquad (1.4.3)$$

Expression of u by (1.3.4) gives

$$E_k = \tfrac{1}{2}mU^2\sin^2(\omega_0 t + \phi) \qquad (1.4.4)$$

The total energy of the system is

$$E = E_p + E_k = \tfrac{1}{2}m\omega_0^2 A^2 \qquad (1.4.5)$$

where use has been made of $s = m\omega_0^2$, $U = \omega_0 A$, and the identity $\sin^2\sigma + \cos^2\sigma = 1$. The total energy can be rewritten in alternate forms,

$$E = \tfrac{1}{2}sA^2 = \tfrac{1}{2}mU^2 \qquad (1.4.6)$$

The total energy is a constant (independent of time) and is equal either to the maximum potential energy (when the mass is at its greatest displacement and is instantaneously at rest) or to the maximum kinetic energy (when the mass passes through its equilibrium position with maximum speed). Since the system was assumed to be free of external forces and not subject to any frictional forces, it is not surprising that the total energy does not change with time.

If all other quantities in the above equations are expressed in MKS units, then E_p, E_k, and E will be in joules (J).

1.5 COMPLEX EXPONENTIAL METHOD OF SOLUTION

Throughout this book, complex quantities will often, but not always, be represented by **boldface** type. One exception is the definition $j \equiv \sqrt{-1}$. We will use the engineering convention of representing the time dependence of oscillatory functions by $\exp(j\omega t)$, rather than the physics convention of $\exp(-i\omega t)$, because of the many close analogies between acoustics and engineering applications. In many cases, consonance between apparently disparate sources can be resolved by making the transformation of j to $-i$. This may in some cases result in an exchange of complex functions from one type to another, but the textual context will usually resolve any ambiguities. Readers unacquainted with complex numbers should refer to Appendixes A2 and A3.

A more general and flexible approach to solving linear differential equations of the form (1.2.5) is to postulate

$$\mathbf{x} = \mathbf{A}e^{\gamma t} \tag{1.5.1}$$

Substitution gives $\gamma^2 = -\omega_0^2$ or $\gamma = \pm j\omega_0$. Thus, the general solution is

$$\mathbf{x} = \mathbf{A}_1 e^{j\omega_0 t} + \mathbf{A}_2 e^{-j\omega_0 t} \tag{1.5.2}$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are to be determined by initial conditions, $\mathbf{x}(0) = x_0$ and $d\mathbf{x}(0)/dt = u_0$. This results in two equations

$$\mathbf{A}_1 + \mathbf{A}_2 = x_0 \qquad \text{and} \qquad \mathbf{A}_1 - \mathbf{A}_2 = u_0/j\omega_0 = -ju_0/\omega_0 \tag{1.5.3}$$

from which

$$\mathbf{A}_1 = \tfrac{1}{2}(x_0 - ju_0/\omega_0) \qquad \text{and} \qquad \mathbf{A}_2 = \tfrac{1}{2}(x_0 + ju_0/\omega_0) \tag{1.5.4}$$

Note that $\mathbf{A}_1$ and $\mathbf{A}_2$ are complex conjugates, so there are really only two constants a and b, where $\mathbf{A}_1 = a - jb$ and $\mathbf{A}_2 = a + jb$. This must be the case since the differential equation is of second order with two independent solutions and, therefore, with two arbitrary constants to be determined by two initial conditions. Substitution of $\mathbf{A}_1$ and $\mathbf{A}_2$ into (1.5.2) yields

$$\mathbf{x} = x_0 \cos\omega_0 t + (u_0/\omega_0)\sin\omega_0 t \tag{1.5.5}$$

which is identical with (1.3.1). Satisfying the initial conditions, which are both real, caused the imaginary part of $\mathbf{x}$ to vanish as an automatic consequence.

In practice it is unnecessary to go through the mathematical steps required to make the imaginary part of the general solution vanish, for *the real part of the complex solution is by itself a complete general solution of the original real differential equation.* Thus, for example, if we express $\mathbf{A}_1 = a_1 + jb_1$ and $\mathbf{A}_2 = a_2 + jb_2$ in (1.5.2) and, before applying initial conditions, take the real part, we have

$$\text{Re}\{\mathbf{x}\} = (a_1 + a_2)\cos\omega_0 t - (b_1 - b_2)\sin\omega_0 t \tag{1.5.6}$$

Now, application of the initial conditions yields $a_1 + a_2 = x_0$ and $b_1 - b_2 = u_0/\omega_0$ so that Re{x} is identical with (1.3.1). Similarly, a complete solution is obtained if the displacement is written in the complex form

$$\mathbf{x} = \mathbf{A}e^{j\omega_0 t} \tag{1.5.7}$$

where $\mathbf{A} = a + jb$, and only the real part is considered,

$$\text{Re}\{\mathbf{x}\} = a\cos\omega_0 t - b\sin\omega_0 t \tag{1.5.8}$$

From the form (1.5.7), which will be used frequently throughout this book, it is particularly easy to obtain the complex speed $\mathbf{u} = d\mathbf{x}/dt$ and the complex acceleration $\mathbf{a} = d\mathbf{u}/dt$ of the mass. The complex speed is

$$\mathbf{u} = j\omega_0 \mathbf{A} e^{j\omega_0 t} = j\omega_0 \mathbf{x} \tag{1.5.9}$$

and the complex acceleration is

$$\mathbf{a} = -\omega_0^2 \mathbf{A} e^{j\omega_0 t} = -\omega_0^2 \mathbf{x} \tag{1.5.10}$$

The expression $\exp(j\omega_0 t)$ may be thought of as a *phasor* of unit length rotating counterclockwise in the complex plane with an angular speed ω_0. Similarly, any complex quantity $\mathbf{A} = a + jb$ may be represented by a phasor of length $A = \sqrt{a^2 + b^2}$, making an angle $\phi = \tan^{-1}(b/a)$ counterclockwise from the positive real axis. Consequently, the product $\mathbf{A}\exp(j\omega_0 t)$ represents a phasor of length A and initial phase angle ϕ rotating in the complex plane with angular speed ω_0 (Fig. 1.5.1). The real part of this rotating phasor (its projection on the real axis) is

$$A\cos(\omega_0 t + \phi) \tag{1.5.11}$$

and varies harmonically with time.

From (1.5.9) we see that differentiation of $\mathbf{x}$ with respect to time gives $\mathbf{u} = j\omega_0 \mathbf{x}$, and hence the phasor representing speed leads that representing displacement by a phase angle of 90°. The projection of this phasor onto the real axis gives the instantaneous speed, the speed amplitude being $\omega_0 A$. Equation (1.5.10) shows that the phasor $\mathbf{a}$ representing the acceleration is out of phase with the displacement phasor by π rad, or 180°. The projection of this phasor onto the real axis gives the instantaneous acceleration, the acceleration amplitude being $\omega_0^2 A$.

It will be the general practice in this textbook to analyze problems by the complex exponential method. The chief advantages of the procedure, as compared with the trigonometric method of solution, are its greater mathematical simplicity and the relative ease with which the phase relationships among the various mechanical and acoustic variables can be determined. However, care must be taken to obtain the *real* part of the complex solution to arrive at the correct physical equation.

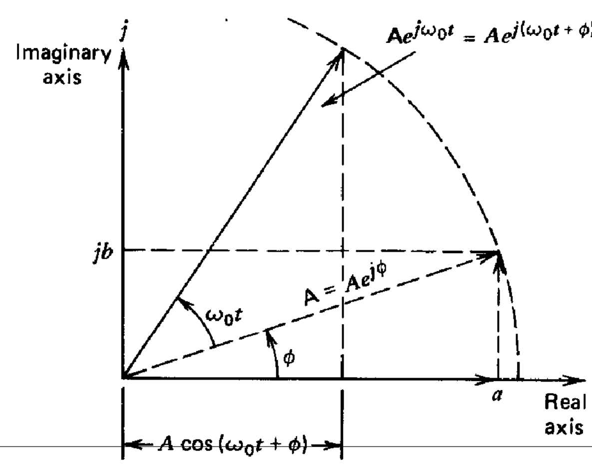


Figure 1.5.1 Physical representation of a phasor $A\exp[j(\omega_0 t + \phi)]$.

1.6 DAMPED OSCILLATIONS

Whenever a real body is set into oscillation, dissipative (frictional) forces arise. These forces are of many types, depending on the particular oscillating system, but they will always result in a *damping* of the oscillations—a decrease in the amplitude of the free oscillations with time. Let us first consider the effect of a *viscous* frictional force f_r on a simple oscillator. Such a force is assumed proportional to the speed of the mass and directed to oppose the motion. It can be expressed as

$$f_r = -R_m \frac{dx}{dt} \tag{1.6.1}$$

where R_m is a positive constant called the *mechanical resistance* of the system. It is evident that mechanical resistance has the units of newton-second per meter (N·s/m) or kilogram per second (kg/s).

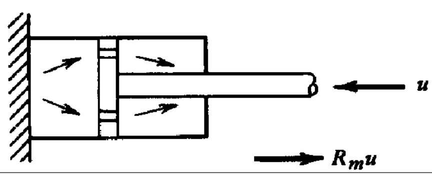
A device that generates such a frictional force can be represented by a dashpot (shock absorber). This system is suggested in Fig. 1.6.1*a*. A simple harmonic oscillator subject to such a frictional force is usually diagrammed as in Fig. 1.6.1*b*.

If the effect of resistance is included, the equation of motion of an oscillator constrained by a stiffness force $-sx$ becomes

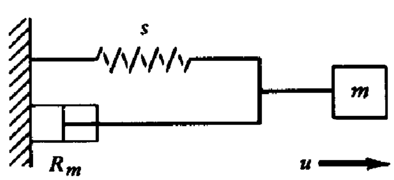
$$m\frac{d^2x}{dt^2} + R_m\frac{dx}{dt} + sx = 0 \tag{1.6.2}$$

Dividing through by m and recalling that $\omega_0 = \sqrt{s/m}$ we have

$$\boxed{\frac{d^2x}{dt^2} + \frac{R_m}{m}\frac{dx}{dt} + \omega_0^2 x = 0} \tag{1.6.3}$$



(*a*)



(*b*)

Figure 1.6.1 (*a*) Representative sketch of a dashpot with mechanical resistance R_m. (*b*) Schematic representation of a damped, free oscillator consisting of a mass m attached to a spring of spring constant s and a dashpot with mechanical resistance R_m.

This equation may be solved by the complex exponential method. Assume a solution of the form

$$\mathbf{x} = \mathbf{A}e^{j\boldsymbol{\gamma} t} \tag{1.6.4}$$

and substitute into (1.6.3) to obtain

$$[\boldsymbol{\gamma}^2 + (R_m/m)\boldsymbol{\gamma} + \omega_0^2]\mathbf{A}e^{j\boldsymbol{\gamma} t} = 0 \tag{1.6.5}$$

Since this must be true for all time,

$$\boldsymbol{\gamma}^2 + (R_m/m)\boldsymbol{\gamma} + \omega_0^2 = 0 \tag{1.6.6}$$

or

$$\boldsymbol{\gamma} = -\beta \pm (\beta^2 - \omega_0^2)^{1/2} \tag{1.6.7}$$

$$\beta = R_m/2m \tag{1.6.8}$$

In most cases of importance in acoustics, the mechanical resistance R_m is small enough so that $\omega_0 > \beta$ and $\boldsymbol{\gamma}$ is complex. Also, notice that if $R_m = 0$ then

$$\boldsymbol{\gamma} = \pm(-\omega_0^2)^{1/2} = \pm j\omega_0 \tag{1.6.9}$$

and the problem has been reduced to that of the undamped oscillator. This suggests defining a new constant ω_d by

$$\omega_d = (\omega_0^2 - \beta^2)^{1/2} \tag{1.6.10}$$

Now, $\boldsymbol{\gamma}$ is given by

$$\boldsymbol{\gamma} = -\beta \pm j\omega_d \tag{1.6.11}$$

and ω_d is seen to be the *natural angular frequency* of the damped oscillator. Note that ω_d is always less than the natural angular frequency ω_0 of the same oscillator without damping.

The complete solution is the sum of the two solutions obtained above,

$$\mathbf{x} = e^{-\beta t}(\mathbf{A}_1 e^{j\omega_d t} + \mathbf{A}_2 e^{-j\omega_d t}) \tag{1.6.12}$$

As in the nondissipative case, the constants $\mathbf{A}_1$ and $\mathbf{A}_2$ are in general complex. As noted earlier, the real part of this complex solution is the complete general solution. One convenient form of this general solution is

$$x = Ae^{-\beta t}\cos(\omega_d t + \phi) \tag{1.6.13}$$

where A and ϕ are real constants determined by the initial conditions. Figure 1.6.2 displays the time history of the displacement of a damped harmonic oscillator for various values of β.

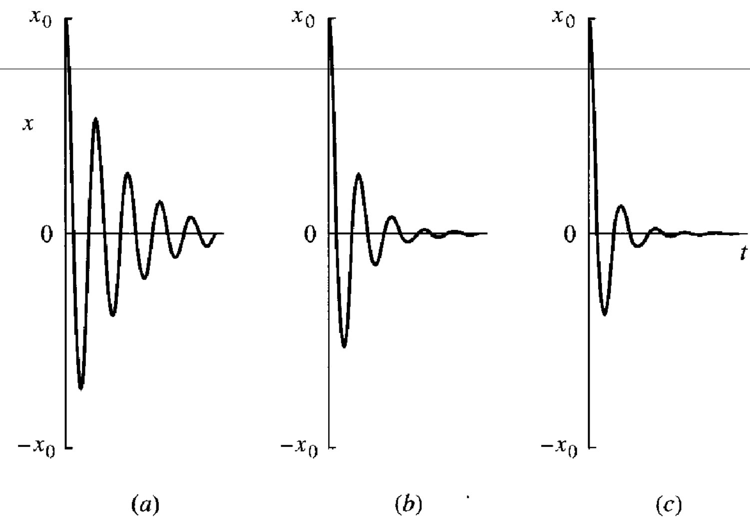


Figure 1.6.2 Decay of an underdamped, free oscillator. Initial conditions: $x_0 = 1$ and $u_0 = 0$. (*a*) $\beta/\omega_0 = 0.1$. (*b*) $\beta/\omega_0 = 0.2$. (*c*) $\beta/\omega_0 = 0.3$.

The amplitude of the damped oscillator, defined as $A\exp(-\beta t)$, is no longer constant but decreases exponentially with time. As with the undamped oscillator, the frequency is independent of the amplitude of oscillation.

One measure of the rapidity with which the oscillations are damped by friction is the time required for the amplitude to decrease to $1/e$ of its initial value. This time τ is the *relaxation time* (other names include *decay modulus, decay time, time constant,* and *characteristic time*) and is given by

$$\tau = 1/\beta = 2m/R_m \tag{1.6.14}$$

The quantity β is the *temporal absorption coefficient.* (As with τ there are a variety of names for β; we mention only one.) The smaller R_m, the larger τ is and the longer it takes for the oscillations to damp out.

If the mechanical resistance R_m is large enough, then $\omega_0 \leq \beta$ and the system is no longer oscillatory; a displaced mass returns asymptotically to its rest position. If $\beta = \omega_0$, the system is known as *critically damped.*

The solution (1.6.13) is the real part of the complex solution

$$\mathbf{x} = \mathbf{A}e^{-\beta t}e^{j\omega_d t} \tag{1.6.15}$$

where $\mathbf{A} = A\exp(j\phi)$. If we rearrange the exponents,

$$\mathbf{x} = \mathbf{A}e^{j(\omega_d + j\beta)t} \tag{1.6.16}$$

we can define a *complex angular frequency*

$$\boldsymbol{\omega}_d = \omega_d + j\beta \tag{1.6.17}$$

whose real part is the angular frequency ω_d of the damped motion and whose imaginary part is the temporal absorption coefficient β. This convention of assimilating the angular frequency and the absorption coefficient into a single complex quantity often proves useful in investigating damped vibrations, as we will see in subsequent chapters.

1.7 FORCED OSCILLATIONS

A simple oscillator, or some equivalent system, is often driven by an *externally applied force* $f(t)$. The differential equation for the motion becomes

$$m\frac{d^2x}{dt^2} + R_m\frac{dx}{dt} + sx = f(t) \tag{1.7.1}$$

Such a system is suggested in Fig 1.7.1.

For the case of a sinusoidal driving force $f(t) = F\cos\omega t$ applied to the oscillator at some initial time, the solution of (1.7.1) is the sum of two parts—a *transient* term containing two arbitrary constants and a *steady-state* term that depends on F and ω but does not contain any arbitrary constants. The transient (homogeneous) term is obtained by setting F equal to zero. Since the resulting equation is identical with (1.6.3), the transient term is given by (1.6.13). Its angular frequency is ω_d. The arbitrary constants are determined by applying the initial conditions to the total solution. After a sufficient time interval $t \gg 1/\beta$, the damping term $\exp(-\beta t)$ makes this portion of the solution negligible, leaving only the steady-state term whose angular frequency ω is that of the driving force.

To obtain the steady-state (particular) solution, it will be advantageous to replace the real driving force $F\cos\omega t$ by its equivalent complex driving force $\mathbf{f} = F\exp(j\omega t)$. The equation then becomes

$$\boxed{m\frac{d^2\mathbf{x}}{dt^2} + R_m\frac{d\mathbf{x}}{dt} + s\mathbf{x} = Fe^{j\omega t}} \tag{1.7.2}$$

The solution of this equation gives the complex displacement $\mathbf{x}$. Since the real part of the complex driving force $\mathbf{f}$ represents the actual driving force $F\cos\omega t$, the real part of the complex displacement will represent the actual displacement.

Because $\mathbf{f} = F\exp(j\omega t)$ is periodic with angular frequency ω, it is plausible to assume that $\mathbf{x}$ must be also. Then, $\mathbf{x} = \mathbf{A}\exp(j\omega t)$, where $\mathbf{A}$ is in general complex. Equation (1.7.2) becomes

$$(-\mathbf{A}\omega^2 m + j\mathbf{A}\omega R_m + \mathbf{A}s)e^{j\omega t} = Fe^{j\omega t} \tag{1.7.3}$$

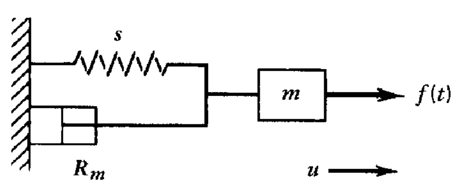


Figure 1.7.1 Schematic representation of a damped, forced oscillator consisting of a mass m driven by a force $f(t)$ attached to a spring of spring constant s and a dashpot with mechanical resistance R_m.

Solving for **A** yields the complex displacement

$$\mathbf{x} = \frac{1}{j\omega}\frac{Fe^{j\omega t}}{R_m + j(\omega m - s/\omega)} \tag{1.7.4}$$

and differentiation gives the complex speed

$$\mathbf{u} = \frac{Fe^{j\omega t}}{R_m + j(\omega m - s/\omega)} \tag{1.7.5}$$

These last two equations can be cast into somewhat simpler form if we define the *complex mechanical input impedance* $\mathbf{Z}_m$ of the system

$$\mathbf{Z}_m = R_m + jX_m \tag{1.7.6}$$

where the *mechanical reactance* X_m is

$$X_m = \omega m - s/\omega \tag{1.7.7}$$

The mechanical impedance $\mathbf{Z}_m = Z_m \exp(j\Theta)$ has magnitude

$$Z_m = [R_m^2 + (\omega m - s/\omega)^2]^{1/2} \tag{1.7.8}$$

and phase angle

$$\Theta = \tan^{-1}(X_m/R_m) = \tan^{-1}[(\omega m - s/\omega)/R_m] \tag{1.7.9}$$

The dimensions of mechanical impedance are the same as those of mechanical resistance and are expressed in the same units, N · s/m, often defined as *mechanical ohms*. It is to be emphasized that, although the mechanical ohm is analogous to the electrical ohm, these two quantities do not have the same units. The electrical ohm has the dimensions of voltage divided by current; the mechanical ohm has the dimensions of force divided by speed.

Using the definition of $\mathbf{Z}_m$ we may write (1.7.5) in the simplified form

$$\boxed{\mathbf{Z}_m = \mathbf{f}/\mathbf{u}} \tag{1.7.10}$$

which gives a most important physical meaning to the complex mechanical impedance: $\mathbf{Z}_m$ is *the ratio of the complex driving force* $\mathbf{f} = F\exp(j\omega t)$ *to the resultant complex speed* $\mathbf{u}$ *of the system at the point where the force is applied.* If, for the driving frequency of interest, the complex impedance $\mathbf{Z}_m$ is known, then we can immediately obtain the complex speed

$$\mathbf{u} = \mathbf{f}/\mathbf{Z}_m \tag{1.7.11}$$

and make use of $\mathbf{u} = j\omega\mathbf{x}$ to obtain the complex displacement

$$\mathbf{x} = \mathbf{f}/j\omega\mathbf{Z}_m \tag{1.7.12}$$

Thus, knowledge of $\mathbf{Z}_m$ is equivalent to solving the differential equation.

The actual displacement is given by the real part of (1.7.4),

$$x = (F/\omega Z_m)\sin(\omega t - \Theta) \tag{1.7.13}$$

and the actual speed is given by the real part of (1.7.5),

$$u = (F/Z_m)\cos(\omega t - \Theta) \tag{1.7.14}$$

[both with the help of (1.7.8) and (1.7.9)]. The ratio F/Z_m gives the maximum speed of the driven oscillator and is the speed amplitude. Equation (1.7.14) shows that Θ is the phase angle between the speed and the driving force. When this angle is positive, it indicates that the speed lags the driving force by Θ. When this angle is negative, it indicates that the speed leads the driving force.

1.8 TRANSIENT RESPONSE OF AN OSCILLATOR

Before continuing the discussion of the simple oscillator it will be well to consider the effect of superimposing the transient response on the steady-state condition. The complete general solution of (1.7.2) is

$$x = Ae^{-\beta t}\cos(\omega_d t + \phi) + (F/\omega Z_m)\sin(\omega t - \Theta) \tag{1.8.1}$$

where A and ϕ are two arbitrary constants whose values are determined by the initial conditions.

As a special case, let us assume that $x_0 = 0$ and $u_0 = 0$ at time $t = 0$ when the driving force is first applied, and that β is small compared to ω_0. Application of these conditions to (1.8.1) gives

$$\begin{aligned} A &= (F/Z_m^2)[(X_m/\omega)^2 + (R_m/\omega_d)^2]^{1/2} \\ \tan\phi &= (\omega/\omega_d)(R_m/X_m) \end{aligned} \tag{1.8.2}$$

Representative curves showing the relative importance of the steady-state and transient terms in producing a combined motion are plotted in Fig. 1.8.1. The effect of the transient is apparent in the left portion of these curves, but near the right end the transient has been so damped that the final steady state is nearly reached. Curves for other initial conditions are analogous, in that the wave form is always somewhat irregular immediately after the application of the driving force, but soon settles into the steady state.

Another important transient is the *decay transient*, which results when the driving force is abruptly removed. The equation of this motion is that of the damped oscillator, (1.6.13), and its angular frequency of oscillation is ω_d not ω. The constants giving the amplitude and phase angle of this motion depend on the part of its cycle in which the driving force is removed. It is impossible to remove the driving force without the appearance of a decay transient, although the effect will be negligible if the amplitude of the driving force is very slowly reduced to zero or the damping is very strong. The decay transient characteristics of mechanical vibrator elements are of particular importance when considering the fidelity of response of sound reproduction components such as loudspeakers and microphones. An example of an overly slow decay is a noticeable

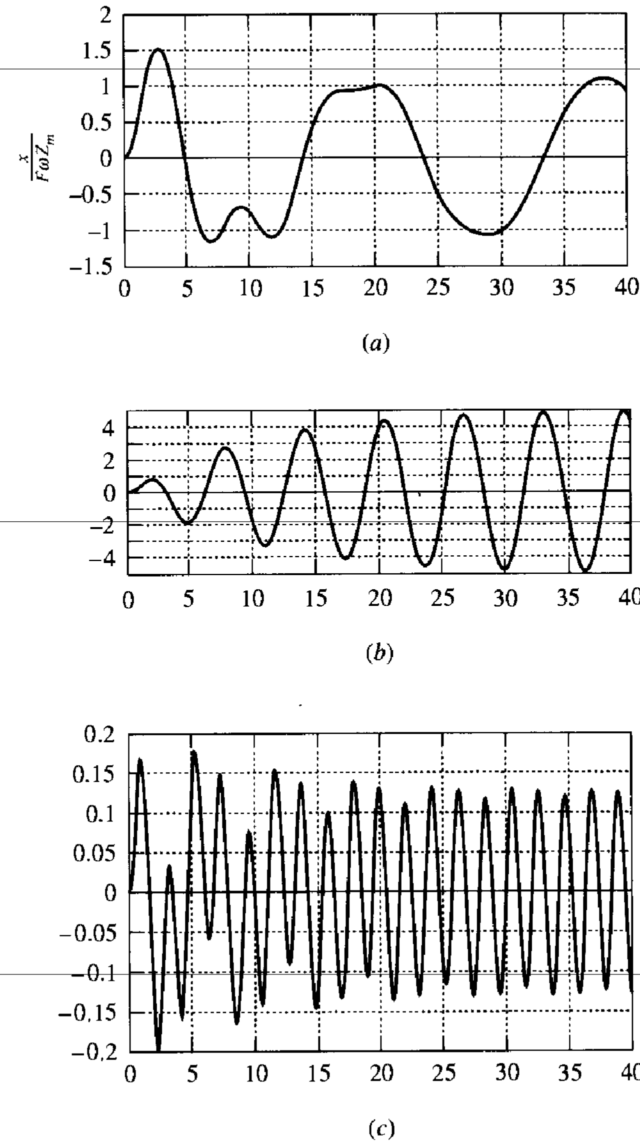


Figure 1.8.1 Transient response of a damped, forced oscillator with $\beta/\omega_d = 0.1$, $x_0 = 0$, and $u_0 = 0$. (*a*) $\omega/\omega_d = \frac{1}{3}$. (*b*) $\omega/\omega_d = 1$. (*c*) $\omega/\omega_d = 3$.

"hangover" at the natural frequency produced by some poorly designed loudspeaker systems.

1.9 POWER RELATIONS

The *instantaneous power* Π_i in watts (W) supplied to the system is equal to the product of the instantaneous driving force and the resulting instantaneous speed. Substituting the appropriate real expressions for the steady-state force and speed,

$$\Pi_i = (F^2/Z_m)\cos\omega t\cos(\omega t - \Theta) \tag{1.9.1}$$

It should be noted that the instantaneous power Π_i is *not* equal to the real part of the product of the complex driving force **f** and the complex speed **u**.

In most situations the *average power* Π being supplied to the system is of more significance than the instantaneous power. This average power is equal to the total work done per complete vibration divided by the time of one vibration,

$$\Pi = \frac{1}{T}\int_0^T \Pi_i \, dt = \langle \Pi_i \rangle_T \tag{1.9.2}$$

Substitution of Π_i in this equation gives

$$\begin{aligned} \Pi &= \frac{F^2}{Z_m T}\int_0^T \cos\omega t \cos(\omega t - \Theta)\, dt \\ &= \frac{F^2}{Z_m T}\int_0^T (\cos^2\omega t \cos\Theta + \cos\omega t \sin\omega t \sin\Theta)\, dt \\ &= \frac{F^2}{2Z_m}\cos\Theta \end{aligned} \tag{1.9.3}$$

This average power supplied to the system by the driving force is not permanently stored in the system but is dissipated in the work expended in moving the system against the frictional force $R_m u$. Since $\cos\Theta = R_m/Z_m$, then (1.9.3) may be written as

$$\Pi = F^2 R_m / 2Z_m^2 \tag{1.9.4}$$

The average power delivered to the oscillator is a maximum when the mechanical reactance X_m vanishes, which from (1.7.7) occurs when $\omega = \omega_0$. At this frequency $\cos\Theta$ has its maximum value of unity ($\Theta = 0$) and Z_m its minimum value R_m.

1.10 MECHANICAL RESONANCE

The *resonance angular frequency* ω_0 is defined as that at which the mechanical reactance X_m vanishes and the mechanical impedance is pure real with its minimum value, $Z_m = R_m$. As has just been noted, at this angular frequency a driving force will supply maximum power to the oscillator. In Section 1.2, ω_0 was found to be the natural angular frequency of a similar undamped oscillator and also the angular frequency of maximum speed amplitude. At $\omega = \omega_0$, (1.7.14) reduces to

$$u_{res} = (F/R_m)\cos\omega_0 t \tag{1.10.1}$$

and the displacement (1.7.13) reduces to

$$x_{res} = (F/\omega_0 R_m)\sin\omega_0 t \tag{1.10.2}$$

(Note that ω_0 does not give the maximum displacement amplitude, which occurs at the angular frequency minimizing the product ωZ_m. It can be shown that this occurs when $\omega = \sqrt{\omega_0^2 - 2\beta^2}$.)

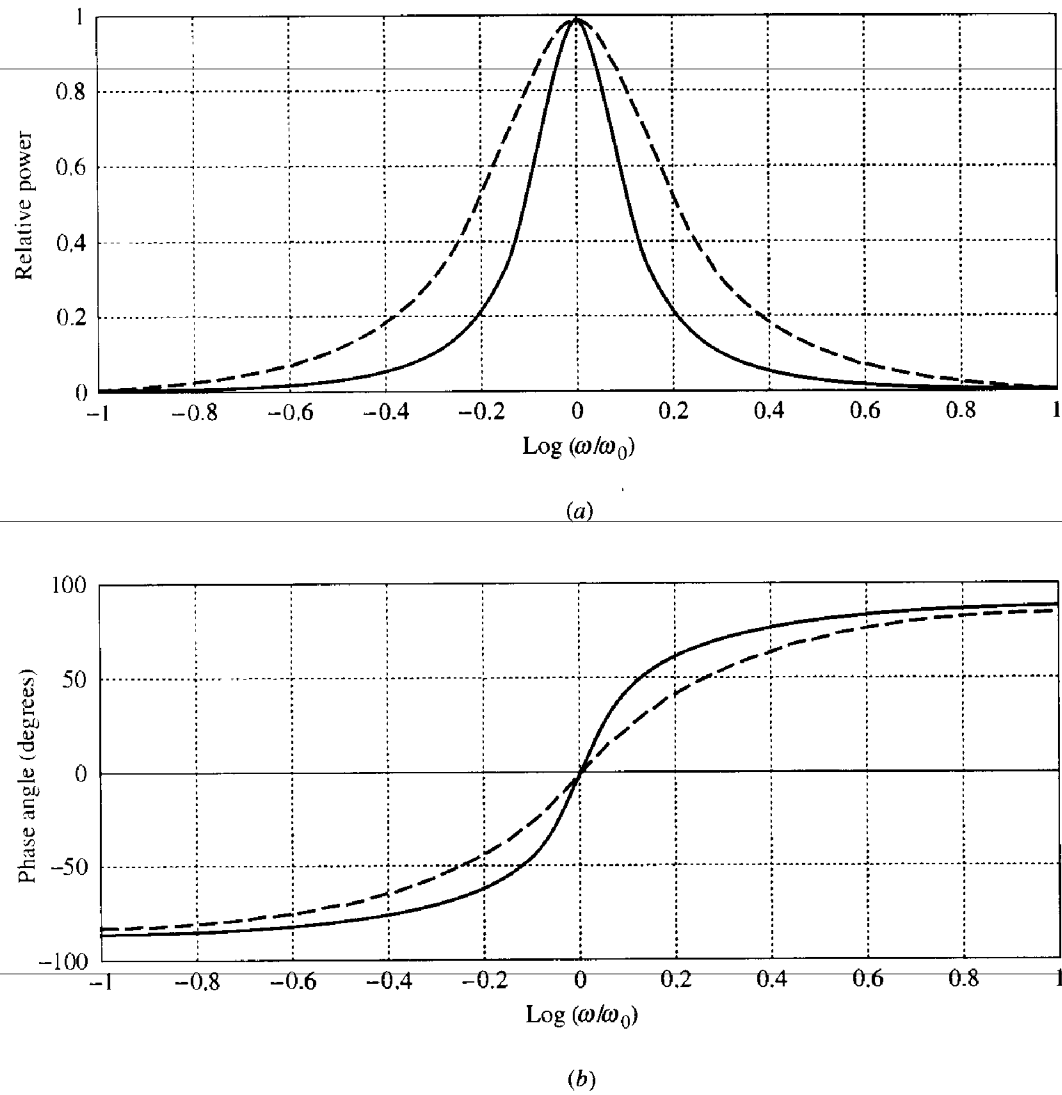


Figure 1.10.1 Response of a simple driven mechanical oscillator. (*a*) Input power relative to its value at resonance. (*b*) Phase angle Θ. Solid lines correspond to $Q = 2$. Dashed lines correspond to $Q = 1$.

If the average power (1.9.4) is plotted as a function of the frequency of a driving force of constant amplitude, a curve similar to Fig. 1.10.1*a* is obtained. It has a maximum value of $F^2/2R_m$ at the resonance frequency and falls at lower and higher frequencies. The sharpness of the peak of the power curve is primarily determined by R_m/m. If this ratio is small, the curve falls off very rapidly—a *sharp resonance.* If, on the other hand, R_m/m is large, the curve falls off more slowly and the system has a *broad resonance.* A more precise definition of the sharpness of resonance can be given in terms of the *quality factor* Q of the system, defined by

$$\boxed{Q = \omega_0/(\omega_u - \omega_l)} \tag{1.10.3}$$

where ω_u and ω_l are the two angular frequencies, above and below resonance, respectively, at which the average power has dropped to one-half its resonance value.

It is also possible to express Q in terms of the mechanical constants of the system. From (1.9.4) it is evident that the average power will be one-half of its resonance value whenever $Z_m^2 = 2R_m^2$. This corresponds to

$$R_m^2 + X_m^2 = 2R_m^2 \quad \text{or} \quad X_m = \pm R_m \tag{1.10.4}$$

Since $X_m = \omega m - s/\omega$, the two values of ω that satisfy this requirement are

$$\omega_u m - s/\omega_u = R_m \quad \text{and} \quad \omega_l m - s/\omega_l = -R_m \tag{1.10.5}$$

The elimination of s between these equations yields

$$\omega_u - \omega_l = R_m/m \tag{1.10.6}$$

so that

$$Q = \omega_0 m/R_m = \omega_0/2\beta \tag{1.10.7}$$

with the help of (1.6.8). Use of (1.6.14) for the relaxation time τ of this oscillator gives

$$Q = \tfrac{1}{2}\omega_0\tau \tag{1.10.8}$$

The sharpness of the resonance of the driven oscillator is directly related to the length of time it takes for the free oscillator to decay to $1/e$ of its initial amplitude. Furthermore, the number of oscillations taken for this decay is $(\omega_d/\omega_0)Q/\pi$ or about Q/π for weak damping. Thus, if an oscillator has a Q of 100 and a natural frequency 1000 Hz, it will take $(100/\pi)$ cycles or 32 ms to decay to $1/e$ of its initial amplitude. It should also be noted that $Q/2\pi$ is the ratio of the mechanical energy of the oscillator driven at its resonance frequency to the energy dissipated per cycle of vibration. Proof of this is left as an exercise (Problem 1.10.3).

When the oscillator is driven at resonance the phase angle Θ is zero and the speed u is in phase with the driving force f. When ω is greater than ω_0 the phase angle is positive, and when ω approaches infinity u lags f by an angle that approaches 90°. When ω is less than ω_0 the phase angle is negative, and as ω approaches zero u leads f by 90°. Figure 1.10.1*b* shows the dependence of Θ on frequency for a typical oscillator. In systems having relatively small mechanical resistance, the phase angles of both speed and displacement vary rapidly in the vicinity of resonance.

1.11 MECHANICAL RESONANCE AND FREQUENCY

Mechanical systems driven by periodic forces can be grouped into three different classes. (1) Sometimes it is desired that the system respond strongly to only *one* particular frequency. If the mechanical resistance of a simple oscillator is small, its impedance will be relatively large at all frequencies except those in the immediate vicinity of resonance, and such an oscillator will consequently respond strongly only in the vicinity of resonance. Some common examples are tuning forks, the resonators below the bars of a xylophone, and magnetostrictive sonar transducers. (2) In other applications it is desired that the system respond strongly to a series of discrete frequencies. The simple oscillator does not have this property, but mechanical systems that do behave in this manner can be designed. These will be considered in subsequent chapters. (3) A third type of use requires that the system respond more or less uniformly to a wide range of frequencies. Examples include the vibrator elements of many electroacoustic and mechanoacoustic transducers: microphones, loudspeakers, hydrophones, many sonar transducers, and the sounding board of a piano.

In different applications, the quantity whose amplitude is supposed to be independent of frequency may be different. In some cases the displacement amplitude is to be independent of frequency; in others it is the speed amplitude or the amplitude of the acceleration that is to be invariant. By a suitable choice of the stiffness, mass, and mechanical resistance, a simple oscillator can be made to satisfy any of these requirements over a limited frequency range. These three special cases of frequency-independent driven oscillators are known as *stiffness-*, *resistance-*, and *mass-controlled* systems, respectively.

A *stiffness-controlled* system is characterized by a large value of s/ω for the frequency range over which the response is to be flat. In this range both ωm and R_m are negligible in comparison with s/m and $\mathbf{Z}_m$ is very nearly equal to $-js/\omega$, so that

$$x \approx (F/s)\cos\omega t \tag{1.11.1}$$

It should be noted that, although the displacement amplitude is independent of frequency, the speed amplitude is not, nor is the acceleration amplitude.

A *resistance-controlled* system is one for which R_m is large in comparison with X_m. This will be true when an oscillator of relatively high mechanical resistance is operated in the vicinity of resonance. Then

$$u \approx (F/R_m)\cos\omega t \tag{1.11.2}$$

so that the speed amplitude is essentially independent of frequency, although both the displacement amplitude and the acceleration are not.

A *mass-controlled* system is characterized by a large value of ωm over the desired frequency range. Then s/ω and R_m are negligible and $\mathbf{Z}_m$ is approximately equal to $j\omega m$. Neither displacement nor speed amplitudes are independent of frequency, but

$$a \approx (F/m)\cos\omega t \tag{1.11.3}$$

so the acceleration amplitude is independent of frequency.

All driven mechanical vibrator elements are resistance-controlled for frequencies nearly equal to their resonant frequency, but for vibrators of low mechanical resistance the range of relatively flat response is extremely narrow. Similarly, all driven vibrators are stiffness-controlled for frequencies well below f_0, and mass-controlled for frequencies well above f_0. A suitable choice of mechanical constants will place any of these systems in the desired part of the frequency range, but the computed values are sometimes very difficult to attain in practice.

*1.12 EQUIVALENT ELECTRICAL CIRCUITS FOR OSCILLATORS

Many vibrating systems are mathematically equivalent to corresponding electrical systems. For example, consider a simple series electrical circuit containing inductance L, resistance R, and capacitance C, driven by an impressed sinusoidal voltage $V \cos \omega t$, as suggested in Fig. 1.12.1*a*. The differential equation for the current $\mathbf{I} = d\mathbf{q}/dt$, where $\mathbf{q}$ is the complex charge, is

$$L\frac{d\mathbf{I}}{dt} + R\mathbf{I} + \frac{\mathbf{q}}{C} = \mathbf{V} \tag{1.12.1}$$

with $\mathbf{V} = V \exp(j\omega t)$. This equation may be written

$$L\frac{d^2\mathbf{q}}{dt^2} + R\frac{d\mathbf{q}}{dt} + \frac{\mathbf{q}}{C} = \mathbf{V} \tag{1.12.2}$$

which has the same form as (1.7.2). Thus, the steady-state solution for $\mathbf{q}$ is

$$\mathbf{q} = \frac{1}{j\omega}\frac{\mathbf{V}}{R + j(\omega L - 1/\omega C)} \tag{1.12.3}$$

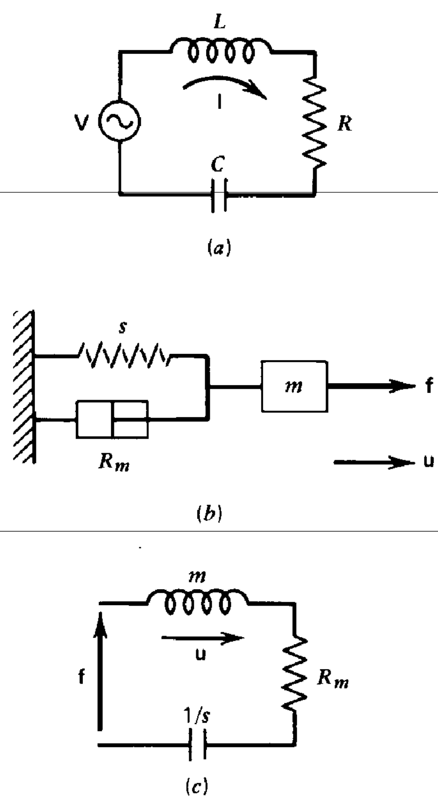


Figure 1.12.1 Equivalent series systems. (*a*) Series electrical circuit driven with voltage $\mathbf{V}$. All elements experience the same current $\mathbf{I}$. (*b*) Mechanical system with mass m driven by force $\mathbf{f}$ and attached to a spring of spring constant s and dashpot of mechanical resistance R_m. All elements move with the same speed $\mathbf{u}$. (*c*) The electrical equivalent of the mechanical system in (*b*).

and the current is $\mathbf{I} = \mathbf{V}/\mathbf{Z}$, where

$$\mathbf{Z} = R + j(\omega L - 1/\omega C) \tag{1.12.4}$$

We see that the electrical circuit of Fig. 1.12.1*a* is the mathematical analog of the damped harmonic oscillator of Fig. 1.12.1*b*. The current **I** in the electrical system is equivalent to the speed **u** in the mechanical system, the charge **q** is equivalent to the displacement **x**, and the applied voltage **V** is equivalent to the applied force **f**. Furthermore, the impedances for these two systems have similar forms, with the mechanical resistance R_m analogous to the electrical resistance R, the mass m analogous to the electrical inductance L, and the mechanical stiffness s analogous to the reciprocal of the electrical capacitance C. By direct comparison of (1.12.1) with (1.7.1), it can be seen that the resonance angular frequency of the electrical circuit is

$$\omega_0 = 1/\sqrt{LC} \tag{1.12.5}$$

and the average power dissipated is

$$\Pi = (V^2/2Z)\cos\Theta \tag{1.12.6}$$

The elements in the electrical system (Fig. 1.12.1*a*) are said to be in *series* because they experience the same current. Similarly the elements in the mechanical system (Fig. 1.12.1*b*) can be represented by the series circuit of Fig. 1.12.1*c*: they experience the same displacement and, therefore, the same speed.

If a simple mechanical oscillator is driven by a sinusoidal force applied to the normally fixed end of the spring as suggested by Fig. 1.12.2*a*, then the mass and the spring experience the same force and this combination is represented by a *parallel* circuit, as shown in Fig. 1.12.2*b*. The speed of the driven end of the spring is equivalent to the current entering the parallel circuit, and the speed $\mathbf{u}_m$ of the mass is equivalent to the current flowing through the inductor.

Other equivalent systems are shown in Figs. 1.12.3 and 1.12.4.

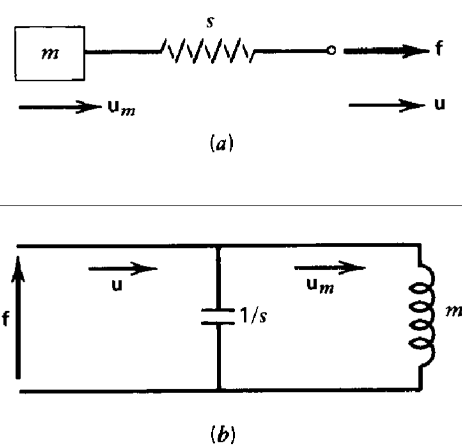
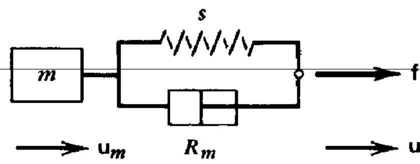
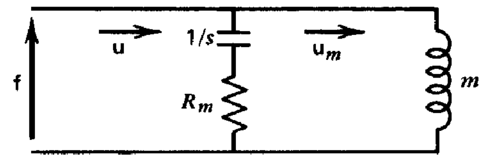


Figure 1.12.2 Equivalent parallel systems. (*a*) Mechanical system with mass attached to a spring and with the other end of the spring driven by a force **f**. The elements feel the same force but have different speeds. (*b*) The equivalent electrical circuit with inductance m and capacitance $1/s$. All elements experience the same voltage but carry different currents.

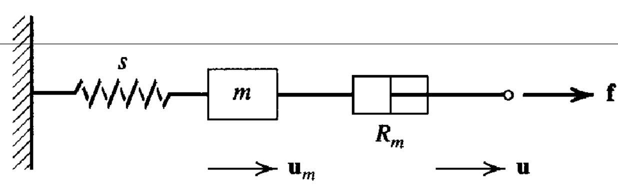


(*a*)

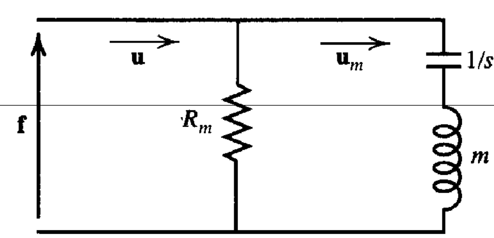


(*b*)

Figure 1.12.3 Equivalent series–parallel systems. (*a*) Mechanical system with mass attached to a combination of spring and dashpot with the other end of the spring/dashpot driven. The dashpot and spring both move with the same speed. They experience different forces, but the sum of forces is equal to the force on the mass. (*b*)The equivalent electrical circuit with inductance, resistance, and capacitance. The capacitance and the resistance share the same current and the sum of the voltages across them equals the voltage across the inductance.



(*a*)



(*b*)

Figure 1.12.4 Equivalent series–parallel systems. (*a*) Mechanical system with mass attached between a spring and a dashpot. One end of the spring is fixed and the dashpot is driven. The mass and spring share the same speed while the sum of forces on them equals the force on the dashpot. (*b*) The equivalent mechanical circuit. The capacitance and the inductance carry the same current and the sum of the voltages across them equals the voltage across the resistance.

1.13 LINEAR COMBINATIONS OF SIMPLE HARMONIC VIBRATIONS

In many important situations that arise in acoustics, the motion of a body is a linear combination of the vibrations induced separately by two or more simple harmonic excitations. It is easy to show that the displacement of the body is then the sum of the individual displacements resulting from each of the harmonic excitations. Combining the effects of individual vibrations by linear addition is valid for the majority of cases encountered in acoustics. In general, the presence of one vibration does not alter the medium to such an extent that the characteristics of other vibrations are disturbed. Consequently, the total vibration is obtained by a *linear superposition* of the individual vibrations.

One case is the combination of two excitations that have the same angular frequency ω. If the two individual displacements are given by

$$\mathbf{x}_1 = A_1 e^{j(\omega t + \phi_1)} \quad \text{and} \quad \mathbf{x}_2 = A_2 e^{j(\omega t + \phi_2)} \tag{1.13.1}$$

their linear combination $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ results in a motion $A \exp[j(\omega t + \phi)]$, where

$$A e^{j(\omega t + \phi)} = (A_1 e^{j\phi_1} + A_2 e^{j\phi_2}) e^{j\omega t} \tag{1.13.2}$$

Solution for A and ϕ can be accomplished easily if the addition of the phasors $A_1 \exp(j\omega t)$ and $A_2 \exp(j\omega t)$ is represented graphically, as in Fig. 1.13.1. From the projections of each phasor on the real and imaginary axes,

$$\begin{aligned} A &= [(A_1 \cos\phi_1 + A_2 \cos\phi_2)^2 + (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2]^{1/2} \\ \tan\phi &= \frac{A_1 \sin\phi_1 + A_2 \sin\phi_2}{A_1 \cos\phi_1 + A_2 \cos\phi_2} \end{aligned} \tag{1.13.3}$$

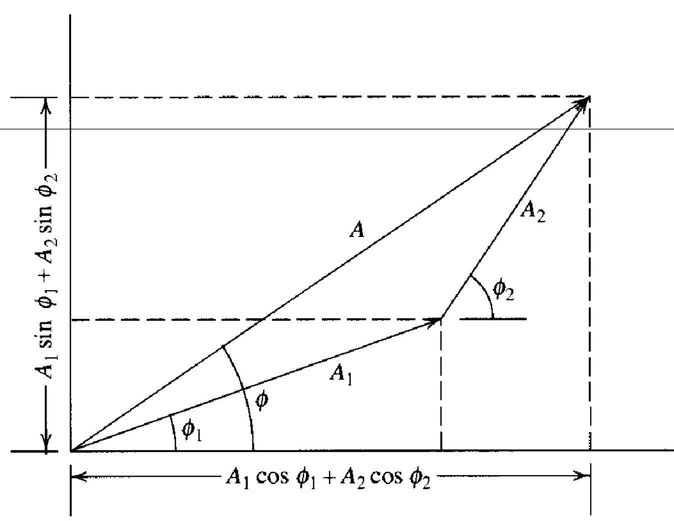


Figure 1.13.1 Phasor combination $A \exp(j\phi) = A_1 \exp(j\phi_1) + A_2 \exp(j\phi_2)$ of two simple harmonic motions having identical frequencies.

The real displacement is

$$x = x_1 + x_2 = A\cos(\omega t + \phi) \tag{1.13.4}$$

where A and ϕ are given by (1.13.3). The linear combination of two simple harmonic vibrations of identical frequency yields another simple harmonic vibration of this same frequency, having a different phase angle and an amplitude in the range $|A_1 - A_2| \le A \le (A_1 + A_2)$.

With the help of Fig. 1.13.1, it is clear that the addition of more than two phasors can be accomplished by drawing them in a chain, head to tail, and then taking their components on the real and imaginary axes. Thus, it may readily be shown that the vibration resulting from the addition of any number n of simple harmonic vibrations of identical frequency has amplitude A and phase angle ϕ given by

$$\begin{aligned} A &= \left[\left(\sum A_n \cos\phi_n\right)^2 + \left(\sum A_n \sin\phi_n\right)^2\right]^{1/2} \\ \tan\phi &= \sum A_n \sin\phi_n \Big/ \sum A_n \cos\phi_n \end{aligned} \tag{1.13.5}$$

Thus, any linear combination of simple harmonic vibrations of identical frequency produces a new simple harmonic vibration of this same frequency. For example, when two or more sound waves overlap in a fluid medium, at each point in the fluid the periodic sound pressures of the individual waves combine as described above.

The expression for the linear combination of two simple harmonic vibrations of *different* angular frequencies ω_1 and ω_2 is

$$\mathbf{x} = A_1 e^{j(\omega_1 t + \phi_1)} + A_2 e^{j(\omega_2 t + \phi_2)} \tag{1.13.6}$$

The resulting motion is not simple harmonic, so that it cannot be represented by a simple sine or cosine function. However, if the ratio of the larger to the smaller frequency is a rational number (commensurate), the motion is periodic with angular frequency given by the greatest common divisor of ω_1 and ω_2. Otherwise, the resulting motion is a nonperiodic oscillation that never repeats itself. The linear combination of three or more simple harmonic vibrations that have different frequencies has characteristics similar to those discussed for two.

The linear combination of two simple harmonic vibrations of nearly the same frequency is easy to interpret. If the angular frequency ω_2 is written as

$$\omega_2 = \omega_1 + \Delta\omega \tag{1.13.7}$$

then the combination is

$$\mathbf{x} = A_1 e^{j(\omega_1 t + \phi_1)} + A_2 e^{j(\omega_1 t + \Delta\omega t + \phi_2)} \tag{1.13.8}$$

This can be reexpressed as

$$\mathbf{x} = (A_1 e^{j\phi_1} + A_2 e^{j(\phi_2 + \Delta\omega t)}) e^{j\omega_1 t} \tag{1.13.9}$$

and then cast into the form

$$\mathbf{x} = Ae^{j(\omega_1 t + \phi)} \tag{1.13.10}$$

where

$$A = [A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_1 - \phi_2 - \Delta\omega t)]^{1/2}$$
$$\tan\phi = \frac{A_1\sin\phi_1 + A_2\sin(\phi_2 + \Delta\omega t)}{A_1\cos\phi_1 + A_2\cos(\phi_2 + \Delta\omega t)} \tag{1.13.11}$$

The resulting vibration may be regarded as *approximately* simple harmonic, with angular frequency ω_1, but with both amplitude A and phase ϕ varying slowly at a frequency of $\Delta\omega/2\pi$. It can be shown that the amplitude of the vibration waxes and wanes between the limits $(A_1 + A_2)$ and $|A_1 - A_2|$. The effect of the variation in phase angle is somewhat more complicated. It modifies the vibration in such a manner that its frequency is not strictly constant, but the average angular frequency may be shown to lie somewhere between ω_1 and ω_2, depending on the relative magnitudes of A_1 and A_2. In the sounding of two pure tones of slightly different frequencies, this variation in amplitude results in a rhythmic pulsing of the loudness of the sound known as *beating*. As an example let us consider the special case $A_1 = A_2$ and $\phi_1 = \phi_2 = 0$. The equations (1.13.11) become

$$A = A_1[2 + 2\cos(\Delta\omega t)]^{1/2}$$
$$\tan\phi = \frac{\sin(\Delta\omega t)}{1 + \cos(\Delta\omega t)} \tag{1.13.12}$$

The amplitude ranges between $2A_1$ and zero, and the beating is very pronounced. Audible beats and other associated phenomena will be discussed in more detail in Chapter 11.

1.14 ANALYSIS OF COMPLEX VIBRATIONS BY FOURIER'S THEOREM

In the preceding section we noted that the linear combination of two or more simple harmonic vibrations with commensurate frequencies leads to a complex vibration that has a frequency determined by the greatest common divisor. Conversely, by means of a powerful mathematical theorem originated by Fourier, it is possible to analyze any complex periodic vibration into a harmonic array of component frequencies.

Stated briefly, this theorem asserts that any single-valued periodic function may be expressed as a summation of simple harmonic terms whose frequencies are integral multiples of the repetition rate of the given function. Since the above restrictions are normally satisfied in the case of the vibrations of material bodies, the theorem is widely used in acoustics.

If a certain vibration of period T is represented by the function $f(t)$, then Fourier's theorem states that $f(t)$ may be represented by the harmonic series

$$f(t) = \tfrac{1}{2}A_0 + A_1 \cos \omega t + A_2 \cos 2\omega t + \cdots + A_n \cos n\omega t + \cdots \\ + B_1 \sin \omega t + B_2 \sin 2\omega t + \cdots + B_n \sin n\omega t + \cdots \tag{1.14.1}$$

where $\omega = 2\pi/T$ and the A's and B's are constants to be determined.

The formulas for evaluating these constants (derived in standard mathematical texts) are

$$A_n = \frac{2}{T}\int_0^T f(t) \cos n\omega t \, dt \\ B_n = \frac{2}{T}\int_0^T f(t) \sin n\omega t \, dt \tag{1.14.2}$$

Whether or not these integrations are feasible will depend on the nature and complexity of the function $f(t)$. If this function exactly represents the combination of a finite number of pure sine and cosine vibrations, the series obtained by computing the above constants will contain only these terms. Analysis, for instance, of simple beats will yield only the two frequencies present. Similarly, the complex vibration constituting the sum of three pure musical tones will analyze into those frequencies alone. On the other hand, if the vibration is characterized by abrupt changes in slope, like sawtooth waves or square waves, then the entire infinite series must be considered for a complete equivalence of motion. If $f(t)$ and df/dt are piecewise continuous over the interval $0 \le t \le T$, it is possible to show that the harmonic series is always convergent. However, jagged functions will require the inclusion of a large number of terms merely to achieve a reasonably good approximation to the original function, and there may be difficulties close to discontinuities. Fortunately, the majority of vibrations encountered in acoustics are relatively smooth functions of time. In such cases, the convergence is rather rapid and only a few terms must be computed.

Depending on the nature of the function being expanded, some terms in the series may be absent. If the function $f(t)$ is symmetrical with respect to $f = 0$, the constant term A_0 will be absent. If the function is *even*, $f(t) = f(-t)$, then all sine terms will be missing. An *odd* function, $f(t) = -f(-t)$, will cause all cosine terms to be absent.

In analyzing the perception of sound, a factor enabling us to reduce the number of higher frequency terms to be computed is that the subjective interpretation of a complex sound vibration is often only slightly altered if the higher frequencies are removed or ignored.

Let us apply the above analysis to a square wave of unit amplitude and period T, defined as

$$f(t) = \begin{cases} +1 & 0 \le t < T/2 \\ -1 & T/2 \le t < T \end{cases} \tag{1.14.3}$$

and repeating every period. Substitution into (1.14.2) yields all $A_n = 0$, $B_n = 0$ for n even, and

$$B_n = 4/n\pi \qquad n = 1, 3, 5, \ldots \tag{1.14.4}$$

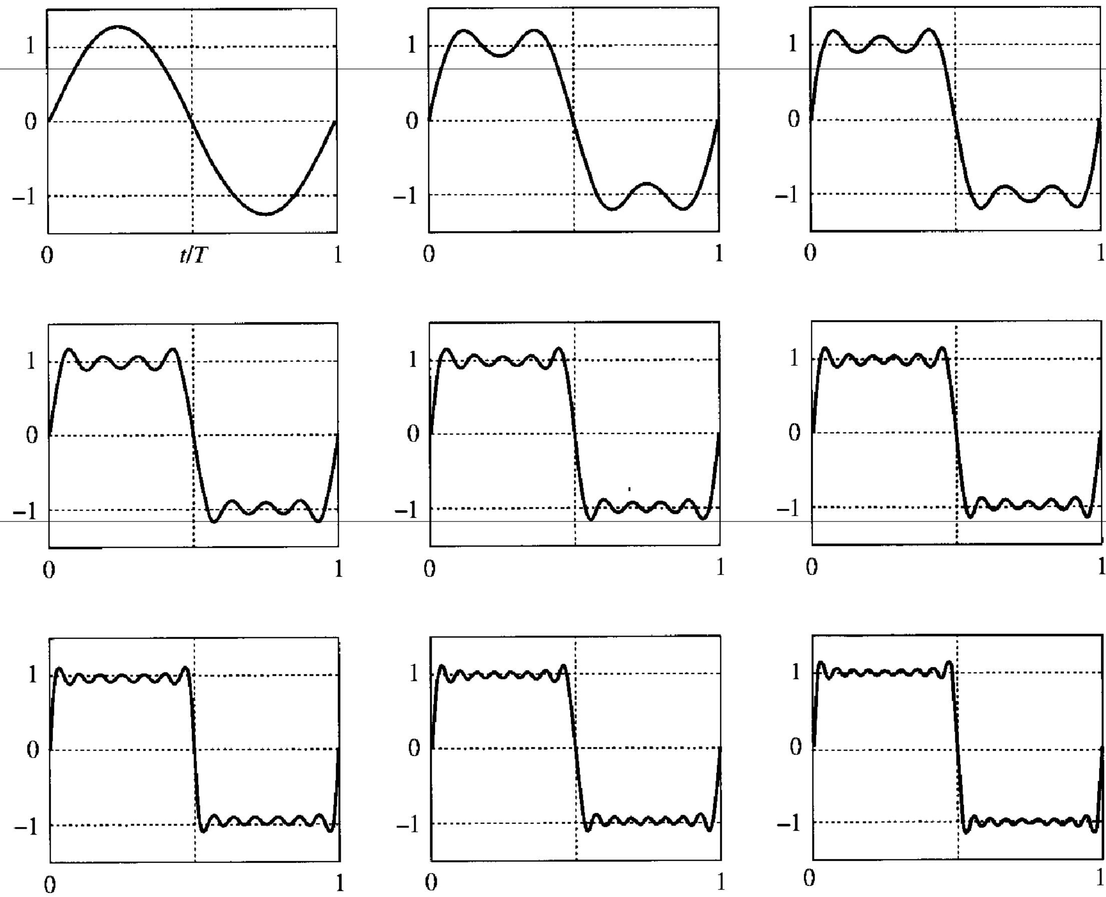


Figure 1.14.1 The Fourier series representation of a square wave vibration of unit amplitude and period T showing the results of including the lowest nonzero harmonics one at a time.

Note that A_0 is zero because of the symmetry of the motion about $f = 0$. All A_n are zero since the function is odd. The B_n are zero for even n because of the symmetry of $f(t)$ within each half-period. The complete harmonic series equivalent to the square wave vibration is

$$f(t) = \frac{4}{\pi}\left(\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \cdots + \frac{1}{n}\sin n\omega t + \cdots\right) \quad (1.14.5)$$

Plotted in Fig. 1.14.1 are results obtained by retaining various numbers of terms of the series. Differences among the plots are quite apparent. Because of the discontinuities, the Fourier series develops visible overshoot near these times if a large enough number of terms are retained.

*1.15 THE FOURIER TRANSFORM

Two fundamental methods are available for the analysis of pulses and other signals of finite duration: the Laplace transform and the Fourier transform. While the Laplace transform is a common approach, the underlying physics is somewhat hidden and there must be no